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# A Class of Local Constant Kernel Estimators for a Regression in a Besov Space

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# A CLASS OF LOCAL CONSTANT KERNEL ESTIMATORS FOR A REGRESSION IN A BESOV SPACE

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**Abstract.** The use of higher order kernels is a well-known method for bias reduction of density and regression estimators. This method of bias reduction has the disadvantage of potential negativity of the underlying estimated density. To avoid this, Mynbaev and Martins-Filho (2010) pioneered a new set of nonparametric kernel based estimators for a density that achieves bias reduction by using a new family of kernels. In addition, Mynbaev and Martins-Filho (2014) obtained much faster convergence of nonparametric prediction by allowing fractional smoothness for the relevant densities. By extending both approaches, in this paper, we propose local constant estimators for regression which are more general than the Nadaraya-Watson (NW) estimator. The main contribution in this paper is that bias reduction may be achieved relative to the NW estimator, and our proposed estimators attain faster uniform convergence without using higher-order kernels and allowing for fractional smoothness for the relevant densities and regressions. We also provide consistency and asymptotic normality of the estimators in the class we propose. A small Monte Carlo study reveals that our estimator performs well relative to the NW estimator and the promised bias reduction is obtained, experimentally in finite samples.

**Keywords and phrases.** bias reduction, local polynomial estimation, asymptotic normality

**JEL classifications.** C13; C14.

**AMS-MS classifications.** 62G07, 62G08, 62G20.

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where the binomial coefficients  $C_{2k}^N = \frac{(2k)!}{(2k-N)!N!}$ ,  $N = 0, \dots, 2k$ ,  $k \geq 1; 2; \dots$  and  $c_{k;s} = (-1)^{s+k} C_{2k}^{s+k}$ ,  $s = -k; \dots; k$ . Mynbaev and Martins-Filho (2014) obtain new results on nonparametric prediction by relaxing the conditions in Carroll et al. (2009)<sup>1</sup> and allowing fractional smoothness of the density. In this paper, by extending the approaches of Mynbaev and Martins-Filho (2010) and Mynbaev and Martins-Filho (2014) we propose a new family of local constant estimators. Based on the kernels  $M_k$  in (3) we define a class of local constant estimators indexed by  $k$  such that

$$\hat{m}_k(x) = \frac{\sum_{t=1}^n M_k\left(\frac{x_t - x}{h_n}\right) Y_t}{\sum_{t=1}^n M_k\left(\frac{x_t - x}{h_n}\right)}. \quad (4)$$

The estimators  $\hat{m}_k(x)$  form a general class of local constant estimators. When  $k = 1$  and a seed kernel  $K$  is symmetric, our estimator  $\hat{m}_1(x)$  coincides with  $\hat{m}(x)$  which is given by (2). That is, the Nadaraya-Watson estimator  $\hat{m}$  can be considered as a special case of our estimators  $\hat{m}_k$ .

Throughout this paper, we assume that the true regression  $m$  belongs to a Besov space  $B_{1,q}^r$  where  $1 \leq q < \infty$  and  $r > 0$ . This assumption is desirable for the following reasons: (i)  $l$ -times continuous differentiability and uniform boundedness of  $m$  is stronger than  $m \in B_{1,q}^r$  where  $l < r$ , that is,  $C^l(\mathbb{R}) \subset B_{1,q}^r$  where  $C^l(\mathbb{R})$  denotes the space of  $l$

Monte Carlo study to investigate the finite sample performance of the local constant estimators we propose and compare it to that of the Nadaraya-Watson estimator using a Gaussian kernel. The simulation results indicate improved performance, measured by the absolute average bias and the the absolute average root mean squared error when the kernels proposed in Mynbaev and Martins-Filho (2010) are used.

The remainder of the paper is organized as follows. Section 2 provides a brief discussion of Besov spaces

where  $c_{k;s} = (-1)^{s+k} C_{2k}^{s+k}$  for  $s = k; \dots; k$  and  $k \geq 1; 2; \dots; g$ . It is easy to verify that for  $s = 2k$ ,

$$\tilde{h}^{-2k} f(x) = \sum_{j=0}^{2k} (-1)^{2k-j} C_{2k}^j f(x+jh) = \sum_{j=1}^k (-1)^{s+k} C_{2k}^{s+k} f(x+kh+sh).$$

Next, we introduce Besov spaces  $B$

which establishes that every  $M_k$ . The kernel  $M_k$  defines a new family of density estimators indexed by  $k$  as follows,

$$\hat{f}_k(\cdot)$$

$k = 1; 2; \dots$ , we have

$$(a) \text{Bias}(\hat{f}_k(x)) = \frac{1}{C_{k,0}} \int_{\mathbb{R}} K(\cdot) \frac{2^k}{h} f(x) dx$$

$$(b) | \text{Bias}(\hat{f}_k(x)) | \leq C h_n^r \int_{\mathbb{R}} |K(\cdot)| \int_{\mathbb{R}} |f^{(r+1)}(x)| dx^{1-q} \int_{\mathbb{R}} |f| dx^{q-1} \text{ where } 2k > r.$$

We note that the order of the bias for our estimator is similar to that attained by the Rosenblatt density estimator constructed with a kernel of order  $r$ . Given Assumption 3 we have  $\text{Bias}(\hat{f}_k(x)) \neq 0$  as  $n \rightarrow \infty$  which implies that  $\hat{f}_k$  is asymptotically unbiased. The following theorem deals with the consistency of  $\hat{f}_k$ .

**Theorem 2** Suppose Assumptions 1, Assumption 2(1)-(2), Assumption 3 and Assumption 4(1)-(4) hold. In addition, suppose that  $\int_{\mathbb{R}} |K(\cdot)| \int_{\mathbb{R}} |f^{(r+1)}(x)| dx^{1-q} < 1$  where  $1-q + 1-q^0 = 1$  for  $1 < q < 1$ . Then, for all  $x \in \mathbb{R}$  and  $k = 1; 2; \dots$ ,

$$\hat{f}_k(x) - f(x) = o_p(1)$$

It is of interest to establish the uniform consistency of  $\hat{f}_k$ . The following theorem provides conditions under which  $\hat{f}_k(x)$  converges to  $f(x)$  uniformly in probability.

**Theorem 3** Suppose Assumption 1, Assumption 2(1)-(2), Assumption 3 and Assumption 4(1)-(5) hold. In addition, suppose that  $\int_{\mathbb{R}} |K(\cdot)| \int_{\mathbb{R}} |f^{(r+1)}(x)| dx^{1-q} < 1$  where  $1-q + 1-q^0 = 1$  for  $1 < q < 1$ . Let  $G$  be a compact subset of  $\mathbb{R}$ . For all  $x \in \mathbb{R}$  and  $k = 1; 2; \dots$ , we have

$$\sup_{x \in G} | \hat{f}_k(x) - f(x) | = O_p \left( \frac{\log n}{nh_n} + h_n^r \right) \quad (12)$$

Uniform consistency of the density estimator requires  $\frac{\log n}{nh_n} \rightarrow 0$  as  $n \rightarrow \infty$ . From (12), the order of  $\hat{f}_k$  is similar to that attained by Rosenblatt density estimator with a kernel of order  $r$ . We achieve much faster uniform convergence rate by imposing the less restrictive assumption  $f \in B_{1,q}^r$ . The next theorem gives the asymptotic normality of the density estimator  $\hat{f}_k(x)$  for all  $x \in \mathbb{R}$  under suitable normalization.

**Theorem 4** Suppose Assumption 1, Assumption 2(1)-(2), Assumption 3 and Assumption 4(1)-(4). Then for all  $x \in \mathbb{R}$  and  $k = 1; 2; \dots$ , we have

$$\sqrt{nh_n} (\hat{f}_k(x) - f(x)) \xrightarrow{D} N(0; f(x) M_k^2(\cdot))$$





literature for (2). We make the following additional assumption.

Assumption 6:  $E\{Y^2 | X\} < \infty$  for  $X \in \mathcal{X}$  and  $\text{Var}(Y | X = x) = \sigma^2(x) < \infty$ .

The estimators  $\hat{m}_k$  are similar to the Nadarya-Watson estimator with the exception that  $K$  is replaced by  $M_k$  kernel. When  $k = 1$  and a seed kernel  $K$  denoted by (9) is symmetric, the estimator  $\hat{m}_1(x)$  coincides with the Nadaraya-Watson estimator (henceforth NW). Thus, the NW estimator is an element of the class

defined in (4). To obtain an approximation to the finite sample properties of

Theorem 6 Suppose Assumption 1, Assumption 2(2), Assumption 3, Assumption 4(1),(4),(5), Assumption 5(2) and Assumption 6 hold. In addition, suppose that  $\frac{nh_n}{\log n} \rightarrow \infty$  as  $n \rightarrow \infty$ . For  $k = 1, 2$ ,

$$\sup_{x \in G} |g_k(x) - E[g_k(x)]| = O_p\left(\frac{\log n}{nh_n}\right)^{1-2k} \quad (15)$$

where  $G$  is a compact set in  $\mathbb{R}$ .

We establish the asymptotic normality of  $g_k(x)$  under a suitable normalization below.

Theorem 7 Suppose Assumption 1-3, Assumption 4(1)-(4), Assumption 6 hold. For  $x \in \mathbb{R}$  and  $k = 1, 2$ ; we have

$$\sqrt{nh_n} [g_k(x) - E(g_k(x)|X_t)] \xrightarrow{d} N(0, \int_0^1 f(x)^2 M_k^2(u) du)$$

Given  $\hat{f}_k(x)$  such that  $\hat{f}_k(x) = f(x) + o_p(1)$

Next theorem states that  $\hat{m}_k(x)$  converges to  $m(x)$  uniformly in probability.

Theorem 9 Suppose Assumption 1-6 hold. In addition, suppose that  $\int_{\mathbb{R}} |K(u)|^q |j^{q^0} j^{(r+1)q^0} d^i|_{1=q^0} < 1$  where  $1=q^0 + 1=q^0 = 1$  for  $1 \leq q \leq 1$ . For  $x \in \mathbb{R}$ ,  $k = 1; 2; \dots$ ;

$$\sup_{x \in G} |\hat{m}_k(x) - m(x)| = O_p \left( h_n^r + \frac{\log n}{nh_n} \right);$$

Uniform consistency of  $\hat{m}_k$  requires  $\frac{\log n}{nh_n} \rightarrow 0$  as  $n \rightarrow \infty$ . We improve the rate of uniform consistency relative to the existing literatures (Devroye (1978), Collomb (1981), Mack and Silverman (1982)) by avoiding higher-order conditions on the kernel and imposing less restrictive conditions.

We now give sufficient condition for asymptotic normality of  $\hat{m}_k(x)$  under suitable centering and normalization.

Theorem 10 Suppose Assumption 1-6 hold. In addition, suppose that  $\int_{\mathbb{R}} |K(u)|^q |j^{q^0} j^{(r+1)q^0} d^i|_{1=q^0} < 1$  where  $1=q^0 + 1=q^0 = 1$  for  $1 \leq q \leq 1$ . For  $x \in \mathbb{R}$  and  $k = 1; 2; \dots$ , we have

$$\sqrt{nh_n} (\hat{m}_k(x) - m(x)) \xrightarrow{D} N(0, \sigma^2)$$

estimator, which is given by  $\hat{m}_{NW}(x) = \hat{m}_1(x) \frac{1}{(nh_n)} \sum_{j=1}^n K\left(\frac{x_j - x}{h_n}\right) Y$

absolute bias (B), average variance (V) and average root MSE (R) of our estimators  $\hat{m}$



Lemma 2

*Proof.* Let  $s$  and  $l \in \mathbb{Z}_+$  such that  $l < r < s$ .

$z$   
 $j_h$



$$\begin{aligned}
&= \sup_{x \in \mathbb{R}} |D^s f(x)|^q \int_0^1 h^{r+sq-1} + h^{r-1} (h)^{sq} dh \\
&+ \sup_{x \in \mathbb{R}} |D^l f(x)|^q \int_0^1 c_1 h^{r+sq-1} + c_2 h^{r-1} (h)^{sq} dh \\
&= \sup_{x \in \mathbb{R}} |D^s f(x)|^q \frac{1}{(s-r)q} (1 + (1)^{sq}) + \sup_{x \in \mathbb{R}} |D^l f(x)|^q \frac{1}{(r-l)q} (c_1 + c_2 (1)^{sq})
\end{aligned}$$

where  $s > r > l$  and some  $c_1, c_2 < 1$ . Therefore, for some  $c_3, c_4$  and  $c < 1$ , we have

$$\int_0^1 |h|^{r-1} |D^s f(x)|^q \frac{dh}{h} \leq c_3 \sup_{x \in \mathbb{R}} |D^s f(x)|^q + c_4 \sup_{x \in \mathbb{R}} |D^l f(x)|^q \leq c \sup_{x \in \mathbb{R}} |D^l f(x)|^q$$

The last inequality follows from the fact that  $C^s(\mathbb{R}) \subset C^l(\mathbb{R})$  for  $s > l$ . Hence, we have  $\|f\|_{B_{r,q}^s} \leq c \|f\|_{B_{r,q}^l}$ .

That is,  $C^l(\mathbb{R}) \subset B_{r,q}^s(\mathbb{R})$  where  $l < r$ .

Theorem 1

Proof. (a)

$$E(\hat{f}_k(x)) = \int \frac{1}{h_n} \frac{1}{c_{k,0}} \int_{|s|=1}^* c_{k,s} K\left(\frac{y-x}{sh_n}\right) f(y) dy = \int \frac{1}{c_{k,0}} \int_{|s|=1}^* c_{k,s} K\left(\frac{y-x}{sh_n}\right) f(x+sh_n) dy$$

Therefore,  $Bias(\hat{f}_k(x))$  can be denoted as follows,

$$Bias(\hat{f}_k(x)) = E(\hat{f}_k(x)) - f(x) = \int \frac{1}{c_{k,0}} K\left(\frac{y-x}{sh_n}\right) f(x+sh_n) dy - f(x)$$

by  $\frac{1}{c_{k,0}} \int_{|s|=1}^* c_{k,s} = 1$  and Assumption 3(2).

(b) We can proceed the order of  $Bias(\hat{f}_k(x))$  using the result of (a).

Given that  $\int_{\mathbb{R}} |K(t)|^q |t|^{(r+1)q} dt < 1$ , we have

$$\|Bias(\hat{f}_k(x))\|_q = \|E(\hat{f}_k(x)) - f(x)\|_q = \left\| \int \frac{1}{c_{k,0}} K\left(\frac{y-x}{sh_n}\right) f(x+sh_n) dy - f(x) \right\|_q$$

Given that

(3962 0 Td [( )] 9.962

$$= \frac{1}{C_{k,0}} \int_{\mathbb{R}} |K(\cdot)|^q j h_n^{(r+1)q} d^{1=q} \int_{\mathbb{R}} \frac{\sup_{x \in \mathbb{R}} |f(x)|^q}{|t|^r} \frac{1}{|t|} h_n^{1=q} dt$$

$$h_n^r \frac{1}{C_{k,0}} \int_{\mathbb{R}} |K(\cdot)|^q j^{(r+1)q} d^{1=q} \int_{\mathbb{R}} |f|^{q^2} dt = O(h_n^r)$$

where  $1=q+1=q^0=1$  and  $1=q-1$ . □

Theorem 2

*Proof.*

$$\begin{aligned} \text{Var}(\hat{f}_k(x)) &= E[\hat{f}_k(x)^2] - (E[\hat{f}_k(x)])^2 \\ &= \int_{\mathbb{R}} \left( \frac{1}{nh_n} \sum_{t=1}^n M_k \left( \frac{y-x}{h_n} \right) f(y) \right)^2 - \left( \int_{\mathbb{R}} \frac{1}{nh_n} \sum_{t=1}^n M_k \left( \frac{y-x}{h_n} \right) f(y) dy \right)^2 \\ &= \int_{\mathbb{R}} \frac{1}{nh_n^2} M_k^2(\cdot) f(x+h_n) d - \frac{1}{n} \int_{\mathbb{R}} M_k(\cdot) f(x+h_n) d^2 \quad \text{given Assumption 1} \\ &= \frac{1}{nh_n} \int_{\mathbb{R}} M_k^2(\cdot) f(x+h_n) d \end{aligned}$$

Now provided that Assumption 2(2), Assumption 3 and Assumption 4(1),(3),(4), we have

$$\begin{aligned} \int_{\mathbb{R}} M_k^2(\cdot) f(x+h) d &= \int_{\mathbb{R}} M_k^2(\cdot) [f(x+h) - f(x)] d + \int_{\mathbb{R}} M_k^2(\cdot) f(x) d \\ \int_{jh}^j M_k^2(\cdot) j f(x+h) - f(x) j d &+ \int_{jh}^j M_k^2(\cdot) j f(x+h) - f(x) j d + f(x) \int_{\mathbb{R}} M_k^2(\cdot) d \\ \sup_{j \in \mathbb{Z}} \int_{x \in \mathbb{R}} j f(x+y) - f(x) j M_k^2(\cdot) d &+ 2 \sup_{x \in \mathbb{R}} \int_{jh}^j j f(x) j M_k^2(\cdot) d + \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} j f(x) j M_k^2(\cdot) d \end{aligned}$$

since  $f \in C^0(\mathbb{R})$  (16)

The inequality follows from  $\int_{\mathbb{R}} M_k^2(\cdot) d \leq C \int_{\mathbb{R}} |K(\cdot)|^q d < 1$  by Assumption 4(3)-(4) for some  $C < 1$  and  $\sup_{x \in \mathbb{R}} |f(x)| < 1$  by Assumption 2(2). If  $h_n \neq 0$  and  $nh_n \neq 1$  as  $n \rightarrow \infty$  (Assumption 3), from Theorem 1 and equation (16),  $\hat{f}_k(x) - f(x) = o_p(1)$  for all  $x \in \mathbb{R}$ . □

Theorem 3

*Proof.* Let  $f_{X_t} g_{t=1,2;\dots,n}$  be a sequence of IID random variables in  $\mathbb{R}$  (Assumption 1). For  $x \in \mathbb{R}$ ,  $\hat{f}_k(x) = \frac{1}{nh_n} \sum_{t=1}^n M_k \left( \frac{X_t - x}{h_n} \right)$  where  $h_n > 0$ . Let  $G$  be a compact subset of  $\mathbb{R}$  that is,  $G \subset \mathbb{R}$ . The

collection  $F = \{B(x; r) : x \in G; r > 0\}$  is an open covering of  $G$ . By the Heine-Borel theorem, the open covering has a finite subcovering. That is, there exists a collection  $F^0 = \{B(x_j; r_j) : x_j \in G; r_j > 0; j = 1, 2, \dots, m\}$  where  $m$  is finite such that  $G \subset \bigcup_{j=1}^m F^0$ . Given that  $K$  satisfies a Lipschitz condition of order 1 Assumption 4(5), for  $x \in G$ , we have

$$\begin{aligned} |f_k(x) - \hat{f}_k(x)| &= \frac{1}{nh_n} \sum_{t=1}^n M_k \frac{|X_t - x|}{h_n} = \frac{1}{nh_n} \sum_{t=1}^n M_k \frac{|X_t - x|}{h_n} \\ &\leq \frac{1}{nh_n} \sum_{t=1}^n \frac{1}{c_{k,0}} \sum_{j=1}^k \frac{c_{k,s_j}}{j s_j} K \frac{|X_t - x|}{sh_n} \end{aligned}$$

Hence,  $P \frac{d_n}{a_n} > M^i$



Hence,  $P \frac{1}{a_n} \max_{1 \leq m \leq n} \hat{f}_k(x)$

$$2r = 2 \frac{h^3}{n} \cdot 1^{1-2} = 2r_n. \text{ Since}$$

$G \cap F^0$  is a covering for  $G$ , it m

$x_0 \in \mathbb{R}$  and  $r_0 < 1$  such that

implies that  $m \cdot r_0 r_n^{1-2} = r_0$

$$2mn^{-4n} = 2r_0$$

$$= 2r_0$$

Since  $nh_n \rightarrow 1$  it suffices to h

Given that  $4_n = M^2 \frac{4N}{(\log)}$

$f(x^m) \in M_k^2(\cdot) d$  as  $n \rightarrow 1$

$nh_n \rightarrow 1$  it suffices to choose

obtain  $n^{\frac{4-n}{4n}} \rightarrow 1$ .

Now,

$$\sup_{x \in G} j \hat{f}_k(x) = f(x) j \sup_{x \in G}$$

$$= \frac{1}{2}$$

#### Theorem 4

*Proof.* We have for  $x \in \mathbb{R}$ ,  $\hat{f}_k(x)$

$$\text{Let } Z_{nt} = \frac{1}{nh_n} M_k \left( \frac{x_t - x}{h_n} \right), E$$

$$S_n^2 = \sum_{t=1}^n E \left[ \frac{1}{nh} M_k \right]$$

$$= \gamma ar$$

of  $B(x; r)$  for  $x \in \mathbb{R}$  is

where  $m$  is finite  $g$  such that

$G$  is bounded, there exists

$$2m \frac{h^3}{n} \cdot 1^{1-2} = 2r_0 \text{ which}$$

$$\frac{1}{n^3 n^{2 \cdot 4_n - 1}} \cdot 1^{1-2}$$

$$\frac{1}{4_n - 1}$$

$$4_n \rightarrow M^2, g_n(x^m) \rightarrow$$

$$\frac{M^2}{f(x^m) M_k^2(\cdot) d} \rightarrow 1. \text{ Since}$$

$$\text{or } \frac{M^2}{2f(x^m) M_k^2(\cdot) d} \rightarrow 2 \text{ to}$$

$$= 2 O_p(1) + h_n^r O(1)$$

□

ii

$\frac{x}{h}$

$x$



since  $\mathbb{R}^j \mathcal{K}(\cdot)j$



by Assumption 4(2) and  $\frac{1}{c_{k,0}} \sum_{j=1}^k c_{k,j} = 1$ .

Next, we prove (b); the order of  $Bias(\hat{g}_k(x))$ .

$$|Bias(\hat{g}_k(x))| = \frac{1}{c_{k,0}} \sum_{j=0}^k c_{k,j} K(\cdot) m(x + sh_n^j) / F11 \quad 9.9626 \quad Tf \quad 3.874 \quad 0 \quad Td \quad [(x)] \quad ]TJ/F8 \quad 9.9626 \quad T$$

Then,

$$\begin{aligned} \text{Var}[\hat{g}_k(x)] &= \frac{1}{nh_n} \int M_k^2(x) f(x+h_n) dx + \frac{1}{n} \int M_k(x) m(x+h_n) f(x+h_n) dx \\ &+ \frac{1}{nh_n} \int M_k^2(x) m^2(x+h_n) f(x+h_n) dx \end{aligned} \quad (21)$$

Since  $f$  and  $m \in C^0(\mathbb{R})$  and Assumption 4(3)-(4), we have

$$\begin{aligned} \int M_k^2(x) f(x+h) dx &\leq \sup_{x \in G} |f(x)| \int M_k^2(x) dx = O(1) \\ \int M_k^2(x) m^2(x+h) f(x+h) dx &\leq \sup_{x \in G} |m(x)|^2 \sup_{x \in G} |f(x)| \int M_k^2(x) dx = O(1) \\ \int M_k(x) m(x+h) f(x+h) dx &\leq \sup_{x \in G} |m(x)| \sup_{x \in G} |f(x)| \int M_k(x) dx = O(1) \end{aligned}$$

Given that  $nh_n \rightarrow 1$  as  $n \rightarrow \infty$ , we have  $\text{Var}[\hat{g}_k(x)] \rightarrow 0$ . Hence  $\hat{g}_k(x) \xrightarrow{p} g(x)$ . □

Theorem 6

*Proof.* Let  $\{X_t\}_{t=1,2,\dots}$  be a sequence of IID random variable in  $\mathbb{R}$ .

$$\hat{g}_k(x) \in \mathbb{R}$$

$$\frac{1}{nh_n} \sum_{t=1}^n M_k \frac{X_t - x}{h_n} = M_k \frac{X_t - x}{h_n} j_m(X_t) j$$

$$\frac{1}{nh_n} \sum_{t=1}^n \frac{1}{c_{k,0}} \sum_{t=1}^n \frac{c_{k,s}}{j|s|} \leq K \frac{X_t - x}{sh_n} = K \frac{X_t - x}{sh_n} j_m(X_t) j$$

by Lipschitz condition on  $K$  (Assumption 4(5)) and  $m \geq C^0(\mathbb{R})$  (Assumption 5(2)).

$$C \sup_{x \in \mathbb{R}} j_m(x) j \frac{|x - x_j|}{h_n^2} = C \sup_{x \in \mathbb{R}} j_m(x) j \frac{r}{h_n^2} \text{ since } x \in B(x_j; r) \text{ which implies } |x - x_j| < r$$

$$\text{and } jE[s_1(x)] - E[s_1(x_j)] j = c \sup_{x \in \mathbb{R}} j_m(x) j \frac{r}{h_n^2}.$$

Thus, from (22) we have

$$j s_1(x) - E[s_1(x)] j = \frac{2cr}{h_n^2} + j s_1(x_j) - E[s_1(x_j)] j:$$

Since for each  $x \in G$ , there exists  $B(x_j; r)$  that contains  $x$ ,

$$d_n = \sup_{x \in \mathbb{R}} j s_1(x) - E[s_1(x)] j = \frac{2cr}{h_n^2} + \max_m j s_1(x_j) - E[s_1(x_j)] j$$

where  $d_n$  is a sequence of stochastic variables. If every  $\epsilon > 0$  there exists  $M > 0$  and a non stochastic sequence  $\{a_n\}$  such that  $P \left\{ \frac{d_n}{a_n} > M \right\} < \epsilon$  for all  $n$ . We write  $d_n = O_p(a_n)$ .

$$\text{Let } d_{2;n} = \max_m j s_1(x)$$

$$\begin{aligned}
jW_{tnj} &= \frac{1}{h_n} M_k \frac{X_t - x}{h_n} m(X_t) - \frac{1}{h_n} E M_k \frac{X_t - x}{h_n} m(X_t) \\
&= \frac{1}{h_n} \frac{1}{c_{k,0}} \sum_{j=1}^k \frac{c_{k,s}}{j^s} K \frac{X_t - x}{sh_n} m(X_t) - \frac{1}{h_n} \frac{1}{c_{k,0}} \sum_{j=1}^k \frac{c_{k,s}}{j^s} E K \frac{X_t - x}{sh_n} m(X_t) \\
&= \frac{1}{h_n} c B_1 \sup_{x \in \mathbb{R}} |jm(x)| + \frac{1}{h_n} \int_{\mathbb{R}} j f(x) dx - 2c B_1 \frac{1}{h_n} \sup_{x \in \mathbb{R}} |jm(x)| \text{ where } c = \frac{1}{c_{k,0}} \sum_{j=1}^k \frac{c_{k,s}}{j^s}.
\end{aligned}$$

since  $\int_{\mathbb{R}} j f(x) dx = 1$ ,  $m \in C^0(\mathbb{R})$  and  $\sup_{x \in \mathbb{R}} |K(x)| \leq B_1$  for all  $x \in \mathbb{R}$ .

$$\begin{aligned}
\text{Var}(W_{tn}) &= E(W_{tn}^2) \\
&= \frac{1}{h_n^2} \int_{\mathbb{R}} M_k^2 \left( \frac{x}{h_n} \right) m(x) f(x) dx - \left( \frac{1}{h_n} \int_{\mathbb{R}} M_k \left( \frac{x}{h_n} \right) m(x) f(x) dx \right)^2 \\
&= \frac{1}{h_n} \int_{\mathbb{R}} M_k^2(x + h_n) m^2(x + h_n) f(x + h_n) dx - \left( \int_{\mathbb{R}} M_k(x + h_n) m(x + h_n) f(x + h_n) dx \right)^2 \\
h_n \text{Var}(W_{tn}) &= \int_{\mathbb{R}} M_k^2(x + h_n) m^2(x + h_n) f(x + h_n) dx - \left( \int_{\mathbb{R}} M_k(x + h_n) m(x + h_n) f(x + h_n) dx \right)^2 \quad (24)
\end{aligned}$$

From (23), we have

$$\begin{aligned}
P[|s_1(x) - E[s_1(x)]| > a_n M_n] &= P\left[ \sum_{t=1}^n W_{tn} > a_n M_n \right] = P\left[ \sum_{t=1}^n W_{tn} > n a_n M_n \right] \\
&= 2 \exp \left( - \frac{a_n^2 M_n^2 n h_n}{2 h_n \text{Var}(W_{tn}) + \frac{2}{3} B_1 c \sup_{x \in \mathbb{R}} |jm(x)| a_n M_n} \right)
\end{aligned}$$

by Bernstein's inequality:

Let  $g_n(x) = h \text{Var}(W_{tn})$ . Then,

$$\begin{aligned}
P\left[ \frac{1}{a_n} \max_m |s_1(x) - E[s_1(x)]| > M_n \right] &= \sum_{m=1}^n \exp \left( - \frac{a_n^2 M_n^2 n h_n}{2 h_n \text{Var}(W_{tn}) + \frac{2}{3} B_1 c \sup_{x \in \mathbb{R}} |jm(x)| a_n M_n} \right) \\
&= 2m \max_m \exp \left( - \frac{a_n^2 M_n^2 n h_n}{2 g_n(x) + \frac{2}{3} B_1 c \sup_{x \in \mathbb{R}} |jm(x)| a_n M_n} \right) \\
&= 2m \exp \left( - \frac{a_n^2 M_n^2 n h_n}{2 g_n(x^m) + \frac{2}{3} B_1 c \sup_{x \in \mathbb{R}} |jm(x)| a_n M_n} \right) \quad (25)
\end{aligned}$$

where  $x^m$  corresponds to the point of the given function such that  $\exp \left( - \frac{a_n^2 M_n^2 n h_n}{2 h_n E[W_{tn}^2] + \frac{2}{3} B_1 c \sup_{x \in \mathbb{R}} |jm(x)| a_n M_n} \right)$  attains its maximum value.

Thus we have

$$g_n(x^m) = \int_{\mathbb{R}} M_k^2(x + h_n) m^2(x + h_n) f(x + h_n) dx - \left( \int_{\mathbb{R}} M_k(x + h_n) m(x + h_n) f(x + h_n) dx \right)^2 \quad (26)$$

Let  $a_n = \frac{\log n}{nh_n}^{1-2}$  and  $r = \frac{h_n^3}{n}^{1-2}$ . We have

$a$





exists  $M > 0$  and a nonstochastic sequence  $f_n g$  such that  $P \stackrel{h}{\sim} \frac{j_n j}{n}$



$$P \left| \frac{1}{n} \sum_{t=1}^n Z_{tn} - E[Z_{tn}] \right| > b_n M_n = P \left| \frac{1}{n} \sum_{t=1}^n Z_{tn} - nb_n M_n \right| > nb_n M_n$$

$$2 \exp \left( - \frac{b_n^2 M_n^2 n h_n}{2 h_n \text{Var}[Z_{tn}] + \frac{2}{3} c B_1 B_n b_n M_n} \right)$$

by Bernstein's inequality. Then,

$$P \frac{1}{b_n} \max_m \left| \hat{s}_2(x^m) - E[\hat{s}_2(x^m)] \right| > M_n$$

$$= 2 \exp \left( - \frac{b_n^2 M_n^2 n h_n}{2 h_n \text{Var}[Z_{tn}] + \frac{2}{3} c B_1 B_n b_n M_n} \right)$$

$$= 2 m \exp \left( - \frac{b_n^2 M_n^2 n h_n}{2 h_n \text{Var}[Z_{tn}] + \frac{2}{3} c B_1 B_n b_n M_n} \right)$$

$$= 2 m \exp \left( - \frac{b_n^2 M_n^2 n h_n}{2 I_n(x^m) + \frac{2}{3} c B_1 B_n b_n M_n} \right) \quad (31)$$

where  $x^m$  corresponds to the point of the given function such that  $\exp \left( - \frac{b_n^2 M_n^2 n h_n}{2 I_n(x^m) + \frac{2}{3} c B_1 B_n b_n M_n} \right)$

Note that

$$\begin{aligned}
 & nh_n(b_n^2 M_n^2) \\
 = & \log n M^2 + \frac{4c^2}{\log n} O_p(1) + \frac{nh_n}{\log n} O(B^{2(1-a)}) - \frac{4c}{(\log n)^{1=2}} O_p(1) \\
 & 2 \frac{nh_n}{\log n} M O(B_n^{1-a}) + 4c \frac{nh_n}{\log n} O(B_n^{1-a})
 \end{aligned}$$

$E[Z_{tn}] = 0$  where  $m(X_t) = E[Y_t | X_t]$ .

$$\begin{aligned} \text{Var}(Z_{tn}) &= E[Z_{tn}^2] = E \left[ \frac{1}{nh_n} M_k \left( \frac{X_t - x}{h_n} \right) (Y_t - m(X_t)) \right]^2 \\ &= \frac{2}{n^2 h_n^2} \int M_k^2 \left( \frac{y - x}{h_n} \right) f(y) dy \end{aligned}$$

Let  $S_n^2 = \sum_{t=1}^n E[Z_{tn}^2]$  and  $X_{tn} = \frac{Z_{tn}}{S_n} = \frac{\frac{1}{nh_n} M_k \left( \frac{X_t - x}{h_n} \right) (Y_t - m(X_t))}{\sqrt{\frac{2}{n^2 h_n^2} \int M_k^2 \left( \frac{y - x}{h_n} \right) f(y) dy}}$ . Then

$$S_n^2 = \sum_{t=1}^n \frac{2}{n^2 h_n^2} \int M_k^2 \left( \frac{y - x}{h_n} \right) f(y) dy = \frac{2}{nh_n^2} \int M_k^2 \left( \frac{y - x}{h_n} \right) f(y) dy$$

By Liapounov's CLT  $\sum_{t=1}^n X_{tn} \xrightarrow{d} N(0;1)$  provided that  $\lim_{n \rightarrow \infty} \sum_{t=1}^n E |X_{tn}|^{2+\epsilon} = 0$  for some  $\epsilon > 0$ . Note that  $|X_{tn}|^2 = \frac{|M_k \left( \frac{X_t - x}{h_n} \right) (Y_t - m(X_t))|^2}{(nh_n)^{1-2} (c(n))^{1-2}}$  with  $c(n) = \frac{2}{h_n} \int M_k^2 \left( \frac{y - x}{h_n} \right) f(y) dy$ .

Therefore,

$$|X_{tn}|^{2+\epsilon} = M_k^2 \left( \frac{X_t - x}{h_n} \right) (Y_t - m(X_t))^{2+\epsilon}$$

According to assumptions that  $\sup_{x \in \mathbb{R}} jK(x)j < 1$ ,  $\int_{\mathbb{R}} jK(x)j dx < 1$  and  $f \in B_{1,q}^r$ , we have

$$\begin{aligned} \int_{\mathbb{R}} jM_k(x)j^{2+} f(x+h_n) dx &= \int_{\mathbb{R}} \frac{1}{C_{k,0}} \sum_{j=1}^{\infty} \frac{C_{k,s}}{j^s} K \frac{1}{s}^{2+} f(x+h_n) dx \\ &\leq C 2^{1+} \sum_{j=1}^{\infty} \frac{C_{k,s}}{j^s} K \frac{1}{s}^{2+} \int_{\mathbb{R}} jf(x+h_n)j dx \quad \text{by } C_r \text{ inequality} \\ &= C 2^{1+} \sum_{j=1}^{\infty} \frac{C_{k,s}}{j^s} K \frac{1}{s}^{2+} \int_{\mathbb{R}} jf(x+h_n)j dx \leq C 2^{1+} \sum_{j=1}^{\infty} j C_{k,s}^{2+} \int_{\mathbb{R}} \sup_{x \in \mathbb{R}} jf(x)j \int_{\mathbb{R}} jK(t)j^{2+} dt < 1 \end{aligned}$$

since  $f \in C^0(\mathbb{R})$  (Assumption 2(2)) and Assumption 4(3)-(4).

Thus,  $\lim_{n \rightarrow \infty} \mathbb{P}_{t=1}^n E jX_{tn}j^{2+} = 0$ . Then,  $\mathbb{P}_{t=1}^n X_{tn} \xrightarrow{d} N(0;1)$  which implies

$$\frac{\mathbb{P}_{t=1}^n \frac{1}{nh_n} M_k \left( \frac{x_t - x}{h_n} \right) [Y_t - m(x_t)]}{\frac{2}{nh_n^2} \int_{\mathbb{R}} M_k^2 \left( \frac{x_t - x}{h_n} \right) f(x_t) dx_t} \xrightarrow{d} N(0;1). \quad \text{Thus, } \mathbb{P}_{t=1}^n [g_k(x) - E(g_k(x)|X_t)] \xrightarrow{d} N(0; \frac{2}{f(x)} M_k^2(x))$$

□

### Theorem 8

*Proof.* For  $x \in \mathbb{R}$ , we have

$$E[\hat{m}_k(x) - m(x)] = E \left[ \frac{\hat{g}_k(x)}{\hat{f}_k(x)} - \frac{g(x)}{f(x)} \right] = \frac{1}{f(x)} E \left[ \frac{\hat{g}_k(x)}{1 + \frac{1}{f(x)} O_p(h_n^r + (nh_n)^{-1/2})} - \frac{g(x)}{f(x)} \right]$$

since  $E[j$

Theorem 9

*Proof.* For  $x \in \mathbb{R}$  and  $k = 1, 2$ , we have

$$\begin{aligned}
 E[m_k(x)] &= E \left[ \frac{\hat{g}_k(x)}{\hat{f}_k(x)} \right] = E \left[ \frac{g_k(x)}{f(x) + O_p(h_n^r + (nh_n)^{-1/2})} \right] \\
 &= \frac{1}{f(x)} E \left[ g_k(x) \left( 1 + \frac{1}{f(x)} O_p(h_n^r + (nh_n)^{-1/2}) \right) \right] \\
 &= \frac{1}{f(x)} E[g_k(x)] + \frac{1}{f(x)^2} E \left[ g_k(x) O_p(h_n^r + (nh_n)^{-1/2}) \right]
 \end{aligned}$$



$\frac{1}{nh_n} [E(\hat{m}_k(x)|X_t) - m(x)]$ . Note that

$$E[\hat{m}_k(x)|X_t] - m(x) = \frac{\frac{1}{nh_n} \sum_{t=1}^n M_k \left( \frac{X_t - x}{h_n} \right) m(X_t)}{\frac{1}{nh_n} \sum_{t=1}^n M_k \left( \frac{X_t - x}{h_n} \right)} - m(x)$$

$$= \frac{1}{\hat{f}_k(x)} \frac{1}{nh_n} \sum_{t=1}^n M_k \left( \frac{X_t - x}{h_n} \right) m(X_t) - m(x) \quad \#$$

"

$$E \left[ \frac{1}{nh_n} \sum_{t=1}^n M_k \left( \frac{X_t - x}{h_n} \right) m(X_t) \right] - m(x)$$

$$= \int \frac{1}{nh_n} \sum_{t=1}^n \frac{1}{c_{k,0}} \sum_{j=1}^k \frac{c_{k,j}}{j!} K \left( \frac{y - x}{sh_n} \right) m(y) f(y) dy - m(x)$$

$$= \int \frac{1}{c_{k,0}} \sum_{j=1}^k \frac{c_{k,j}}{j!} K \left( \frac{y - x}{sh_n} \right) m(y) f(y) dy - m(x)$$

Consequently,

$$\begin{aligned} P \frac{1}{nh_n} (\hat{m}_k(x) - m(x)) &= P \frac{1}{nh_n} (\hat{m}_k(x) - E[\hat{m}_k(x)|X_t]) + P \frac{1}{nh_n} (E[\hat{m}_k(x)|X_t] - m(x)) \\ &= P \frac{1}{nh_n} (\hat{m}_k(x) - E[\hat{m}_k(x)|X_t]) + P \frac{1}{nh_n} O_p(h_n^r) = P \frac{1}{nh_n} (\hat{m}_k(x) - E[\hat{m}_k(x)|X_t]) + O_p(h^r) \\ &\stackrel{Z}{=} \int_{N(0, \sigma^2 f(x))}^{-1} M_k^2(x) dx \end{aligned}$$

If  $\sigma^2 f(x) > 0$  a.s. we have  $P \frac{1}{nh_n} (\hat{m}_k(x) - m(x)) \stackrel{Z}{=} \int_{N(0, \sigma^2 f(x))}^{-1} M_k^2(x) dx$ . □

## Appendix 2 - Tables and figures

Table 1

Local constant estimator with cross validation bandwidth  $h^{CV}$ ; Trimmed average absolute Bias ( $B$ );  
Trimmed average Variance ( $V$ ); Trimmed average Root Mean Squared Error ( $R$ ).

Methods	$m_1(x)$			$m_2(x)$		
	$BV$	$VR$	$RB$	$BV$	$VR$	$R$
$\hat{m}_{NW}$	0.0517	0.0470	0.2320	0.0384	0.0315	0.0940

[(n)]TJ/29 408.416 Td [(n)]TJ/F8 9.

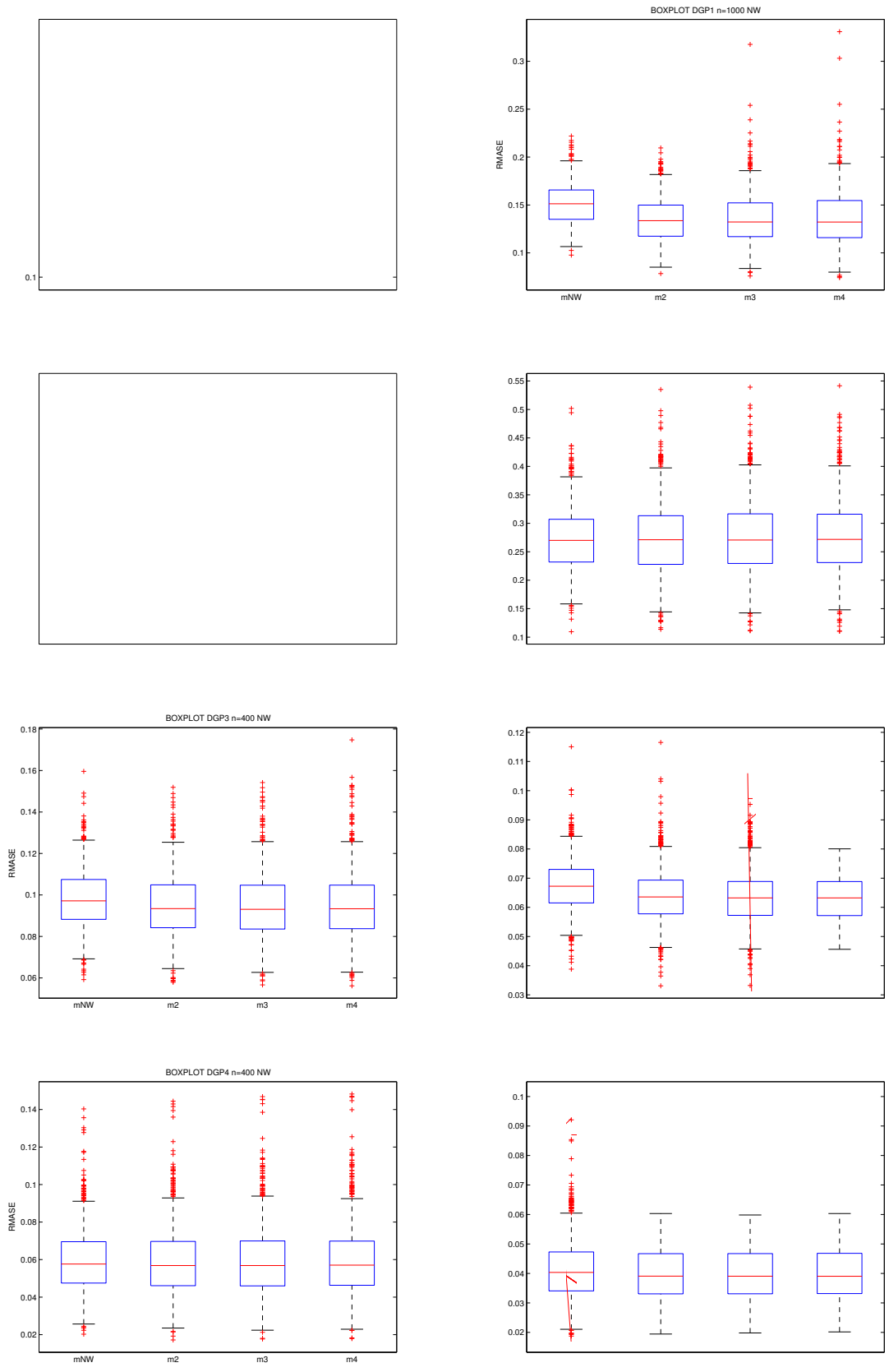
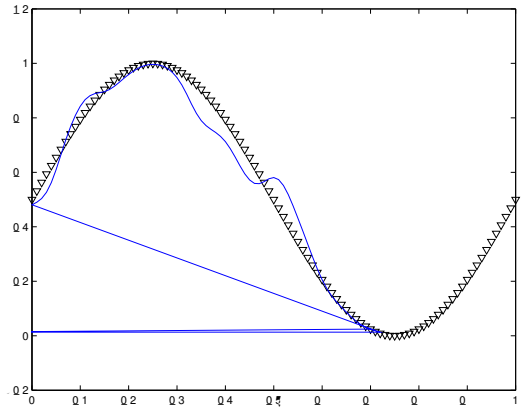
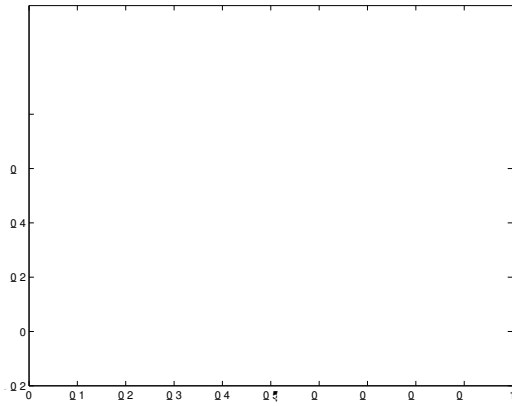
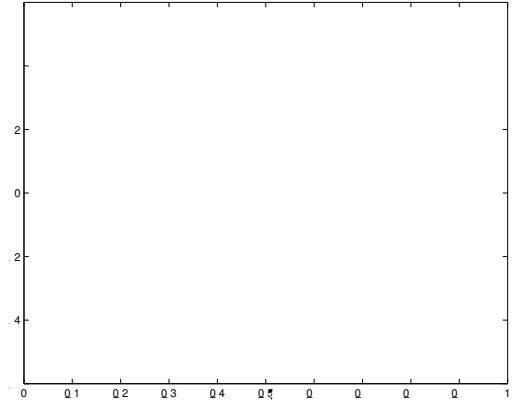
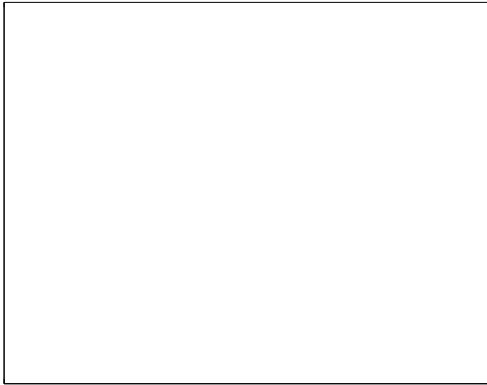


Figure 1: These figures are box plots of trimmed RMSE from estimators  $\hat{m}_{NW}$ ;  $\hat{m}_2$ ;  $\hat{m}_3$  and  $\hat{m}_4$  and four DGPs. DGP1, DGP2, DGP3 and DGP4 indicate  $m_1(x)$ ,  $m_2(x)$ ,  $m_3(x)$  and  $m_4(x)$  respectively.





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