DISCUSSION PAPERS IN ECONOMICS

Working Paper No. 03-13

Modeling Network Externalities, Network Effects, and Product Compatibility with Logit Demand

(Revision of Working Paper No. 03-06)

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November 2003

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November 11, 2003

Abstract

With the advent of the digital age, information goods characterized by strong positive network externalities and effects are playing an increasingly prominent economic role. A logit model of oligopolistic competition is presented with a focus on providing an accessible rigorous analytic framework for positive network externalities and effects.

In the presence of positive network externalities and effects, market behavior is quite different from that of traditional logit models. Multiple stable equilibria arise. Oligopoly producers respond to higher price elasticities with lower prices and markups. Markets tend to be highly concentrated and the dominant producer can remain dominant even while producing an inferior product. Strategic behaviors arise that do not exist in the absence of network externalities or effects.

JEL classification: C65; D11; D43; L13

Keywords: Discrete Choice, Network Externalities, Network Effects

Contents

1 Introduction

As our economy moves into the digital age, products exhibiting network eﬀects and externalities are playing an increasingly important economic role.

Network externalities are externalities that a consumer sees as a result of others' use of a product and similar or compatible products. If eBay[1] had a

[1]eBay is a popular web site (

single user, that user would find little use for the site. And it becomes more useful as more people use the site.

Another example, which is familiar to anyone who uses a computer in their day-to-day lives, is software. While Microsoft Word is intrinsically useful as a word processor, it also provides its users with benefits in the form of network externalities. The more users of Microsoft Word there are, the greater the ability for each user to send documents as e-mail attachments or to otherwise communicate via documents generated in Microsoft Word.

Network effects are externalities that arise from use of a network of compatible products. Microsoft's dominance in the personal computer operating system market was widely attributed to an "applications barrier to entry," a network effect in which a disproportionate number of applications are produced for the dominant operating system, which in turn makes the operating system more desirable.

Network externalities and effects can be either positive or negative, though

spawned a wealth of literature. In a seminal modeling effort, Katz and Shapiro [8] introduced a model of network externalities in which a continuum of consumers make a discrete choice between the products of Cournot

Although a number of stylized models have been advanced, no framework has been established suitable for ready applied examination of information goods that exhibit network externalities or effects. To that end, a discrete choice random utility model of positive network externalities is presented with an emphasis on providing an analytic framework for the economic analysis of these markets.

The model introduces two innovations in a traditional logit setting: A utility function in which a consumer explicitly values the consumption of a product and compatible products by others and a production function with a compatibility decision and an associated cost of compatibility.[5]

In the presence of network externalities, multiple equilibria can arise. However, they can be easily identified and characterized numerically. Although

As a result of network externalities, producers compete in both price and network size and firms use compatibility decisions to strategically react to large, entrenched competitors. Markets are shown to be more responsive to the actions of both dominant and fringe producers. A dominant producer's tendency to extract monopoly profits is mitigated by the need to support a dominant network.

In section 2, a model of network externalities is presented. Numerical results from several scenarios are presented in section 3 and in section 4 the results

The consumer optimization problem will be introduced in section 2.1, followed by the profit maximization problem of the producers in section 2.2. Then section 2.3 will provide first-order conditions for equilibrium and de-

$$u_{i,n} = y + q_i - \gamma p_i + v(z_{i,n}) + \sigma \epsilon_{i,n} \quad (1)$$

where $i \in \{1, ..., I\}$ denotes product i, n denotes consumer n where $n \in \{1, ..., N\}$, $u_{i,n}$ is the utility of product i for consumer n, y is consumer income, q_i is the perceived quality of product i, p_i is the price of good i with γ its elasticity parameter,[8] σ is a scaling parameter corresponding to the degree of heterogeneity across products, and $\epsilon_{i,n}$ is the consumer's idiosyncratic valuation of product i.

Consumer n's perception of the value of the network of product i, v, is taken to be a continuous and strictly increasing function of consumer n's perceived network size, $z_{i,n}$, of product i, that is, others' consumption of product i and compatible products. It is also given that $v(0) = 0$. Compatibility between products i and j is given by $\rho_{i,j}$, where $\rho_{i,j} \in [0, 1]$ and $\rho_{i,j} \frac{\partial z_{i,m}}{\partial x_{i,n}} = \frac{\partial z_{i,m}}{\partial x_{j,n}} \forall i \neq j, m \neq n$. In other words, the parameter $\rho_{i,j}$ describes the impact on the size of the network of additional expected consumption of a compatible

[8]Notably, in [17], Saha and Simon apply a utility function that is polynomial in price to the analysis of mergers and find that the linear specification tends to over-estimate the post-merger price effect.

product, j, relative to additional expected consumption of product i. $\rho_{i,j} = 0$ represents complete incompatibility whereas $\rho_{i,j} = 1$ implies that products i and j are fully compatible.[9]

As we shall see in section 2.2, $\rho_{i,j}$ is determined by compatibility parameters $\varphi_{i,j}$ and $\varphi_{j,i}$, which reflect producers' compatibility decisions.

It is not uncommon to include an outside good, representing a numeraire, in the traditional logit model to represent the choice "none of the above." Good I can serve as the outside good by assuming a price of zero, unitary intrinsic utility, no associated network externalities, and full incompatibility with all other products.

Implicit in the consumers' preferences are strongly additive[10] positive network externalities in which a consumer sees no network externalities unless at least some of a product is consumed by others. Also implicit in the preferences is the traditional logit formulation as the special case in which none

[9]See appendix B for examples of a variety of functional forms expressing consumer

of the products exhibit network externalities.[11]

The base quality can be considered the utility the consumer receives as a result of intrinsic attributes of the product. For example, the user of a word processor gains usefulness from the product by its ability to compose documents. The additional network effects are derived exclusively from the user's ability to interact with other users. Continuing with the word processor example, this may include the ability to send and receive documents[12] to and from other users of the same or compatible word processors and the ability to draw on the knowledge base of other word processor users to accomplish complex tasks. Note that some products may have no base quality. If there were a single fax machine in the world, it wouldn't be doing anybody much good.

The formulation, in concert with the producer decision outlined in section 2.2, will be referred to throughout as the Network MNL (Network Multinomial Logit) Model. Although the treatment of the model for the purposes of this paper is in the context of network externalities, the model would readily

[11] That is, $v_{i,n} \equiv 0 \quad i, n$.

[12] As e-mail attachments, for example.

apply to network effects with little modification.[13]

Based on the utility specification in equation 1, associated with each consumer k and product i is a probability $P_n(i)$ where

$$P_n(i) = P(u_{i,n} = \max_{j=1,\dots,I} u_{j,n}) \tag{2}$$

A further symmetry assumption, $P_m(i) = P_n(i) \quad m, n \quad \{1, \dots, N\}$, is imposed on $P_n(i)$ to provide both analytic and computational tractability which allows us to abbreviate $v_{i,n}$ as v_i and $P_n(i)$ as $P(i)$.[14]

$$P(i) = \Psi_i(x; p, q, \rho) = \frac{e^{q_i - \gamma p_i + v_i(x)}}{\sum_{j=1}^{I} e^{q_j - \gamma p_j + v_j(x)}} \tag{3}$$

Equilibrium is given to be a Nash equilibrium; that is, in equilibrium, consumption decisions are made simultaneously taking prices, product compatibility, and other consumers' choices as given. In equilibrium, $x_i = NP(i)$.

As with the traditional logit demand system, it is easy to show that an equilibrium exists.[16] Unlike the traditional logit demand system, due to the increasing returns inherent in positive network externalities, multiple equilibria can exist; indeed, they are to be expected as a fundamental characteristic of the system when the value of network externalities is sufficiently large and convex in perceived network size.

However, contrary to what one might expect, even in the presence of convex positive network externalities, multiple equilibria are not guaranteed. With weak network externalities and sufficient differentiation between products in terms of core attributes and/or pricing, a single stable equilibria will

[16] See appendix C.1.

be found. More precisely, consumers can be supposed to follow a discrete tatônnement process in which, in each time period, they make their consumption decisions taking prices, compatibility, and network sizes based on the previous time period's choices as given. When for all possible allocations x at prices p, qualities q, and compatibility ρ,

$$\frac{du_i}{dx_i} < \frac{1}{2\Psi_i(x)(1 - \Psi_i(x))} \quad i \tag{4}$$

the solution set reduces to a single, stable equilibrium.[17] In general, for any producer which does not sport a price or feature advantage, network externalities must be strong enough for a large network size to overcome the intrinsic disadvantage in that producer's product attributes or pricing to enjoy a dominant equilibrium.

Given multiple equilibrium, it is typically easy to establish the stability of the equilibrium. On the basis of the tatônnement process, it can be shown[18]

[17] See appendix C.2.

[18] For a more detailed overview of stability, see appendix C.3.

that an equilibrium x^* is stable if

$$\left(\frac{du_i}{dx_i}\right)_{x^*} < \frac{1}{2\Psi_i(x^*)(1 - \Psi_i(x^*))} \quad \forall i \tag{5}$$

Although the stability condition is not also sufficient, it provides a simple means by which the stability of any given equilibrium can be evaluated. From equations 4 and 5, if an equilibrium is unique, it is also stable.

2.2 Producers

Production of good i involves a fixed cost, a cost associated with the level of product quality, a constant marginal cost, and a compatibility cost associated with making a product compatible with other competing products. Compatibility is not assumed to be an equivalence relation; that is, if product i is compatible with product j, it does not imply that product j is equally compatible with product i. In this sense, a compatibility decision can involve construction of either a one-way or two-way adapter or something in between.

continuous function ρ in which $\rho_{i,j}$ denotes $\rho(\varphi_{i,j}, \varphi_{j,i})$.[19] It is assumed that ρ is strictly increasing in $\varphi_{i,j}$ and nondecreasing in $\varphi_{j,i}$, concave in its arguments, and that $\rho(0,0) = 0$ and $\lim_{\varphi_{i,j} \to \infty} \frac{\partial \rho_{i,j}}{\partial \varphi_{i,j}} = \lim_{\varphi_{j,i} \to \infty} \frac{\partial \rho_{i,j}}{\partial \varphi_{j,i}} = 0$. Thus, with no spending on compatibility, products are fully incompatible, and producers experience diminishing marginal compatibility.

While a number of authors have explored the use of both one-way[20] and two-way adapters,[21] the specification of ρ in this exposition assumes that each producer can to some extent control the degree of compatibility of their own product with respect to other products; however, each producer's compatibility decision may impact the relative compatibility of other products. In other words, the functional form of ρ allows for both one-way and two-way adapters and two-way adapters do not necessarily impart equivalent compatibility both ways.

In fact two-way adapters are quite possible97 adax15Tdompatbenefit to compatibility by either producer (as would arise when the industry is not concentrated into a single dominant firm and a competitive fringe). Firms may also use side payments or strategic agreements to enhance compatibility, a strategy that is common in practice,[22] though that possibility is not explicitly considered in the following exposition.

When two-way adapters are not considered, analysis of a producer's decision-making process is somewhat simpler as the introduction of compatibility is of benefit solely to the producer that incurs the costs and to the detriment of all of that producer's competitors. When two-way adapters are allowed, the constraints on the derivatives of ρ imply that a producer i's decision to increase to the compatibility of its product with product j may result in the increase in the effective network size or marginal contribution to network size of compatibility of producer i's competitors.

[22] Microsoft, for example, has a long-standing agreement to provide AOL with a desktop icon in exchange for using Internet Explorer as the built-in browser for the AOL client [13].

2.3 Equilibrium

Equilibrium results from a simultaneous move Bertrand-Nash game. Producers and consumers form expectations regarding consumers' choices with complete information about the consumers' response functions.[23] Producers simultaneously choose price, product quality, and compatibility to maximize profit with complete information about the consumers' response functions. Consumers simultaneously maximize utility by choosing consumption taking prices, product quality, and compatibility as given. In equilibrium, both producers' and consumers' expectations of network size are realized; that is, expectations are rational.

The question of how consumers and producers form their expectations and why there may be a focus on one equilibrium over any other may be based on the problem under consideration. When small exogenous shocks are considered, market players may expect that the resultant equilibrium following an exogenous shock will occur an equilibrium connected by a continuous

[23] That is, producers and consumers form expectations regarding the size of product networks in response to price and compatibility choices.

"path" of stable equilibria to the current equilibria.[24] Alternatively, perhaps consumers expect change will be minimal or the equilibrium that leaves the most dominant players in the most dominant positions may be selected.[25] Another reasonable assertion would be that the equilibrium expectation is formed in a Stackleberg manner by the dominant producer. Whatever the coordination process, it is reasonable to assume that the set of admissible equilibria are stable as defined by a linearization about the equilibrium point of the tatônnement process outlined in appendix C.

First-order conditions for profit maximization in prices are given by

$$x_i + (p_i - b_i)\frac{dx_i}{dp_i} \leq 0 \qquad p_i \geq 0 \tag{7}$$

where, from equation 3, firms face an own-price demand derivative of

[24] If one exists, that is.
[25] Again, if one exists.

$$\frac{dx_i}{dp_i} = -\gamma N P(i)(1 - P(i))[1 + \varepsilon_{p_i}] \tag{8}$$

ε_{p_i} reflects the first-order impact of price changes on network size[26] and is given by

$$\varepsilon_{p_i} = e_i \left(\sum_{n=1}^{\infty} J^n \right) D_p e_i^T \tag{9}$$

where

$$J = \left[\left(\frac{N-1}{N} \right) \sum_{i=1}^{I} \frac{\partial \Psi_m}{\partial v_i} \frac{\partial v_i}{\partial x_n} \right] \tag{10}$$

and

[26] When consumers face a choice set without network externalities, the own-price demand elasticity of product i is given by $\eta_{p_i} = \gamma p_i (1 - P(i))$.

$$D_p = \left(\frac{\partial \Psi_m}{\partial p_n} \right) \tag{11}$$

From equation 8, the own-price elasticity of demand is given by[27]

$$\eta_{p_i} = \gamma p_i (1 - P(i))[1 + \varepsilon_{p_i}] \tag{12}$$

In general, positive network externalities exacerbate consumers' price responses, often quite dramatically. With preferences convex in network size, producers anticipate that consumers are more responsive to changing prices or product attributes than they would otherwise be when considering products that do not exhibit network externalities.

In a regime with no product compatibility, the responsiveness is greatest as is the cost in terms of network benefits of switching from a dominant to a fringe product. With full compatibility, as the number of consumers becomes

[27] See appendix B for details regarding the derivation of 7

large, the price elasticity approaches that found in a market without network externalities as any loss in network size is made up by a corresponding gain in the network size of compatible products, resulting in no net impact to the effective network size.[28] More generally, with full compatibility, as the where the term $\varepsilon_{p_{i,j}}$ reflects the first-order impact to demand for good i of price changes to good j on network size.[29]

As with the expression for own-price demand elasticity, the network term in equation 13 has the greatest impact in the absence of any product compatibility. Without product compatibility, consumers are wary of switching from a product with a strong network to one without. With full product compatibility, as in the case of own-price elasticity of demand, as the number of consumers becomes large, the cross-price elasticity approaches that of a market without network externalities.

The well-known IIA property of logit demand implies that [30] that the cross-price elasticity of demand is equal for any product i with respect to a product j. However, with the introduction of network externalities, it is easy to see from equations 13 and 14 that in the absence of full compatibility, IIA does not necessarily hold in the presence of network externalities. As a result, the familiar behavior associated with the traditional logit model in the presence

[29] In the absence of network externalities, the cross-price elasticity of demand is given by $\eta_{p_{i,j}} = \gamma p_j P(i)$.

[30] See [7, pp. 86–87] and [19, pp. 23–24 and 43–44].

ε_{q_i} reflects the first-order impact of quality changes on network size[31] and is given by

$$\varepsilon_{q_i} = e_i \left(\sum_{n=1}^{\infty} J^n \right) D_q e_i^T \qquad (17)$$

where

D_q

with

$$\varepsilon_{q_{i,j}} = e_i \left(\sum_{n=1}^{\infty} J^n \right) D_q e_j^T \tag{20}$$

where the term $\varepsilon_{q_{i,j}}$ reflects the first-order impact to demand for good i of price changes to good j on network size.[32]

In addition to price and product quality, consumers respond to changes in network size. The first-order conditions for profit-maximization in compatibility for product i with respect to product j, $j = i$, is given by

$(p_i$

$$\frac{dx_i}{d\varphi_{i,j}} = NP(i)P(j) \; \square \; \frac{\partial v_i}{\partial \rho_{i,j}} \frac{\partial \rho_{i,j}}{\partial \varphi_{i,j}}$$

where

$$D_{\varphi_i} = \left(\frac{\partial \Psi_m}{\partial v_i} \frac{\partial v_i}{\partial \rho_{i,n}} \frac{\partial \rho_{i,n}}{\partial \varphi_{i,n}} + \frac{\partial \Psi_m}{\partial v_n} \frac{\partial v_n}{\partial \rho_{n,i}} \frac{\partial \rho_{n,i}}{\partial \varphi_{i,n}} \right) \tag{25}$$

2.4 Calibration

The MNL is known as a "rough and ready" model [7] for the ease with which existing market data can be calibrated against the demand specification and counterfactuals introduced to analyze relevant policy decisions.

The Network MNL is no different in this regard, but involves additional steps to calibrate the scale of the network externalities or effects and incorporate costs of compatibility. While preferences are estimable by well-established econometric techniques and prices and market shares are typically readily observable, compatibility levels may not be. The means by which compatibility levels would be determined would likely be product-specific. For the purposes

Given that the number of consumers, preferences, prices, product qualities, market shares, and compatibilities are observable,[34] and given a suitable functional form for the network term v, by equation 3, a system of simultaneous equations can be solved to yield the scale of the network externalities and from the scale parameterize the function v. Likewise, given a suitable functional form for compatibility ρ and a set of compatibility levels, a system of equations can be solved to yield the compatibility activity of each producer φ.[35]

Given that the number of consumers, preferences, prices, product qualities, market shares, and compatibilities are observable and given a suitable functional form for v, by expression 6, the implied marginal costs borne by a profit-maximizing producer are given by

$$b_i = p_i + x_i / \frac{dx_i}{dp_i}; \qquad (26)$$

[34] Alternatively, elasticities may be used in place of market shares as a primitive of the model. Economists are more accustomed to working with elasticities and readily accessible econometric tools are available for their estimation.

[35] From a practical standpoint, if the matrix of compatibility levels is sparse, parame-
=

or, alternatively,

$$b_i = p_i(1 - \eta_i) \tag{27}$$

with $\frac{dx_i}{dp_i}$ given by equation 8 and η_i given in equation 12. Typically, the dominant producer faces a higher elasticity of demand than the competitive fringe; however, due to the market concentration it enjoys, the calibration marginal costs reflects lower marginal costs (and higher margins and profits) for the dominant producer than for the fringe.

Quality costs can be recovered from the expression

$$(p_i - b_i)\frac{dx_i}{dq_i} - \frac{da}{dq_i} = 0 \tag{28}$$

with $\frac{dx_i}{dq_i}$ given by equation 16.

When the observed compatibility is positive, the implied marginal costs borne by producer i relative to compatibility with product j are given by

$$c_{i,j} = (p_i - b_i)\frac{dx_i}{d\varphi_{i,j}} \tag{29}$$

with $\frac{dx_i}{d\varphi_{i,j}}$ given by equation 22. It is not atypical that some products may be wholly incompatible, in which case a cost of compatibility must be extrapolated from reasonable assumptions and observed calibration costs with

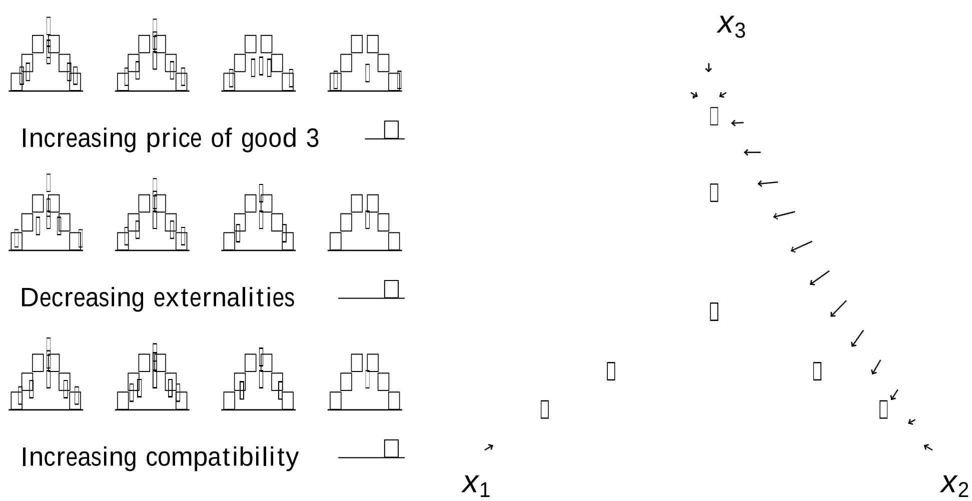
When there are sufficiently strong, convex network externalities, the relative importance of the differentiating features of a product are subsumed by the need to standardize on one of the available choices. The user of a word processor may not care so much that an embedded spreadsheet is dynamically updated as they do that others can read their documents. Furthermore, contrary to traditional models of consumer demand, the dominant good may not even be the preferred good. In fact, in the presence of sufficiently strong network externalities, consumers can rationally choose to standardize on any of the available goods[37] and an equilibrium may exist where the features of the fringe producers' goods may be strongly preferred to the dominant good.[38]

With symmetric preferences convex in network size, the most preferred equilibria arise concentrating demand on any of the I goods. A single least preferred equilibrium arises when demand is concentrated equally on all I goods. Between these stable equilibria, there may exist saddle points in which several producers form a dominant set from which a small perturbation will

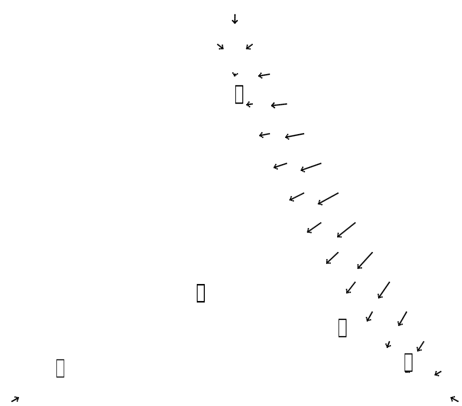
[37] Though due to the distribution of the random utility component ϵ_k there will always be a nonzero probability of choosing any given good.

[38] Much to the chagrin of the fringe producers, no doubt.

push the market toward a regime with a single dominant producer. Overall, consumers prefer equilibria that represent a more concentrated market.



ties is a very satisfied (and profitable) producer. Much more so than in a comparable industry that does not exhibit network externalities. The fringe producers, on the other hand, face a much harsher business climate than they would in an industry without network externalities.



differences in prices or feature sets become increasingly disparate, equilibria dynamic may give rise to counterintuitive strategic behavior in which fringe producers collude to drop prices below the individual profit-maximizing levels in the presence of a strongly dominant competitor or a dominant competitor seeks control of a fringe producer to raise the fringe producer's price to the detriment of other fringe producers.

$\eta_{i,j}$	$j = 1$	$j = 2$	$j = 3$	$\eta_{i,j}$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	−1.98	0.99	0.99	$i = 1$	−1.17	0.58	0.58
$i = 2$	31.84	−25.99	−5.86	$i = 2$	18.83	−19.42	0.58
$i = 3$	31.84	−5.86	−25.99	$i = 3$	18.83	0.58	−19.42

Table 1: Price elasticities with and without network exter-4rice elasticities with and without

4 Discussion

Product nesting [19, section 2.7.1] provides modelers with the ability to accommodate a more diverse prior with regards to elasticities as well as formally model sequential decision-making processes. The Network MNL would benefit from similar extension, allowing complex behaviors, such as the choice of computer operating system in which consumers first choose the set of applications they would like to use then choose the operating system which supports those applications, to be effectively modeled. Bearing in mind the caveat of Chou and Shy [4], nesting would also allow modelers to separate network effects and externalities from complementarities with such products masquerading as network effects and externalities.

In addition to product nesting, incorporation of an outside good representing "none of the above" and multiple consumer types have seen wide application. Multiple consumer types would allow for a richer modeling framework in which, for example, the preferences of tech-savvy "first adopters" of technology products could be separated from more casual users or business use could be separated from recreational use of information goods at home.

The introduction of an intertemporal framework would allow a rich set of strategic behaviors to be considered. With a fixed cost of production, entry and exit decisions can be effectively modeled. And the implications of changes to cross-price and cross-quality derivatives detailed in section 3 would give rise to complex strategic interactions.

Finally, the model would benefit from the introduction of uncertainty, particularly as it may impact decision making surrounding regime changes or other discontinuities in equilibria sets.

The traditional MNL demand formulation has seen wide use in theoretical analysis of differentiated products industries, in simulation analysis of the impact of mergers and acquisitions, including in support of antitrust litigation, 84Td[(Ferro)1i1(lM)1(nTdeen)28(titrust)-355i4435n4mgslysisareTJ94(e□)-areTJroatlyi7latm

A Calculation of Choice Probabilities

Suppose N consumers derive utility over I products of the form

$$u_{i,n} = y_n + q_i + v_{i,n} + \sigma \ _{i,n} \tag{30}$$

where y_n is consumer income, q_i is the intrinsic quality of product i, $v_{i,n}$ is the utility consumer n derives from others' consumption of product i and compatible products, $\ _{i,n}$ is a random variable representing consumer n's idiosyncratic valuation of product i, and σ is a scaling parameter determining the degree of heterogeneity across products.

For the utility of good i to exceed that of good j for consumer n, we need

$$q_i + v_{i,n} + \sigma \ _{i,n} > q_j + v_{j,n} + \sigma \ _{j,n} \tag{31}$$

In other words, we need

$$P(\sigma_{i,n} = \urcorner)P(\urcorner$$

$$
\begin{aligned}
P_n(i) &= \int_{-\infty}^{\infty} \frac{1}{\sigma} e^{-(\frac{\epsilon}{\sigma})} e^{-e^{-(\frac{\epsilon}{\sigma})}} \prod_{j=i} e^{-e^{-(\frac{\epsilon}{\sigma})} e^{-\left(\frac{q_j - q_i + v_{j,n} - v_{i,n}}{\sigma}\right)}} d\epsilon \\
&= -\int_0^{\infty} g(\varphi) \prod_{j=i} e^{-\varphi e^{-\left(\frac{q_j - q_i + v_{j,n} - v_{i,n}}{\sigma}\right)}} dg \\
&= -\int_0^{\infty} g(\varphi) e^{-\varphi e^{-\left(\frac{q_i + v_{i,n}}{\sum_{j \neq i} q_j + v_{j,n}}\right)}} dg \\
&= -\int_0^{\infty} e^{-\varphi\left(1 + \frac{\sum_{j \neq i} q_{j,n} + v_{j,n}}{q_{i,n} + v_{i,n}}\right)} dg \\
&= -\int_0^{\infty} e^{-\varphi\left(\frac{\sum_j q_{j,n} + v_{j,n}}{q_{i,n} + v_{i,n}}\right)} dg \\
&= -\frac{e^{q_{i,n} + v_{i,n}}}{\sum_j e^{q_{j,n} + v_{j,n}}} \left[e^{-\varphi\left(\frac{\sum_j e^{q_{j,n} + v_{j,n}}}{q_{i,n} + v_{i,n}}\right)} \right]_0^{\infty} \\
&= -\frac{e^{q_{i,n} + v_{i,n}}}{\sum_j e^{q_{j,n} + v_{j,n}}} [0 - 1] \\
&= \frac{e^{q_{i,n} + v_{i,n}}}{\sum_j e^{q_{j,n} + v_{j,n}}}
\end{aligned}
$$

The equivalent derivation in the presence the symmetry assumptions of section 2.1 can be performed by dropping the subscript n.

B Calculation of Demand Elasticities

Suppose utility for consumer $n \in \{1, ..., N\}$ for product $i \in \{1, ..., I\}$ based on intrinsic quality q, prices p, and compatibilities ρ is given by

$$u_{i,k} = q_i - \gamma p_i + v(z_{i,n}) + \sigma \epsilon_n \tag{34}$$

where ϵ_n is a random variable distributed type 1 extreme and $z_{i,n}$ is the network size of product i from the perspective of consumer n.

For example, if $z_{i,n}$ is taken to be a function of absolute network size,[40] it would be given by

$$z_{i,n} = \sum_{m=n} (x_{i,m} + \sum_{j=i} \rho_{i,j} x_{j,m}) \tag{35}$$

[40] That is, it is the sum of the expected consumption of product i and all compatible products by all other consumers based on the degree of compatibility between product i and other products

On the other hand, if consumers perceptions are in terms of relative network size, or effective market share, $z_{i,n}$ would be given by

$$z_{i,n} = \frac{\sum_{m=n}(x_{i,m} + \sum_{j=i} \rho_{i,j} x_{j,m})}{\sum_{m=n} x_{i,m}} \tag{36}$$

Assuming symmetric choice probabilities, a closed-form expression for the probability that any given consumer will choose product i is given by

$$P_i = \Psi_i(x) = \frac{e^{q_i - \gamma p_i + v(z_i)}}{\sum_{j=1}^{I} e^{q_j - \gamma p_j + v(z_j)}} \tag{37}$$

and demand for product i is given by $x_i = NP_i$.

From equation 37, the positive network externalities intrinsic to demand produce positive feedback effects in which an external shock (e.g., a change in price) affects demand both through its direct effect on demand and its indirect effect on demand through the network term.

The indirect effect on demand can be characterized by the Jacobian

$$J = \left[\sum_{k=1}^{n} \frac{\partial \Psi_i}{\partial v_k} \frac{\partial v_k}{\partial x_j} \right] \tag{38}$$

where v_k denotes the network term associated with product k.

From 37, the first term of the Jacobian can be decomposed as

$$\frac{\partial \Psi_i}{\partial v_k} = \begin{cases} P_i (1 - P_i) & \text{when } i = k \\ - P_i P_k & \text{otherwise} \end{cases} \tag{39}$$

When absolute network size is assumed,[41] the second term in the Jacobian is given by

[41] That is, that network size is given by equation 35.

$$\frac{\partial v_k}{\partial x_j} = \begin{cases} \frac{N-1}{N}\frac{\partial v_k}{\partial z_j} & \text{when } j = k \\ \rho_{k,j} \quad \frac{N-1}{N}\frac{\partial v_k}{\partial z_j} & \text{otherwise} \end{cases} \tag{40}$$

where the term $(N-1)/N$ arises because consumers only derive value from the use of the same or compatible products by others.

On the other hand, when relative network size is assumed,[42] under the symmetry assumptions the term $\frac{N-1}{N}$ drops out. If the effective market share of product j is given by

$$S_j = \frac{x_j + \sum_{k \neq j} \rho_{j,k} x_k}{\sum_{k=1}^{I} x_k} \tag{41}$$

then $\frac{\partial v_k}{\partial x_j}$ is given by

[42] That is, that network size is given by equation 36.

$$\frac{\partial v_k}{\partial x_j} = \begin{cases} \frac{1}{\sum_{i=1}^{I} x_i} \frac{\partial v_k}{\partial z_j} & \text{when } j = k \\ \frac{\rho_{k,j} - S_j}{\sum_{i=1}^{I} x_i} \frac{\partial v_k}{\partial z_j} & \text{otherwise} \end{cases} \tag{42}$$

The direct effect on demand x_i from a change in price p_j is given by the matrix

$$D_p = \left[\frac{\partial x_m}{\partial p_n} \right] \tag{43}$$

From equation 37, the terms of D_p can be decomposed as

$$\frac{\partial x_i}{\partial p_j} = \begin{cases} - \quad \gamma N P_i (1 - P_i) & \text{when } i = j \\ \gamma N P_i P_j & \text{otherwise} \end{cases} \tag{44}$$

From the effects 38 and 44, the total derivative of demand with respect to price is given by[43]

[43] The discerning reader will recognize $(I - J)$ as a Markov matrix.

$$\frac{dx_i}{dp_j} = e_i \left(J^0 + J^1 + \ldots \right) D_p e_j^T \tag{45}$$

$$= e_i (I - J)^{-1} D_p e_j^T \tag{46}$$

where e_i is the row vector $[0 \ldots 0\ 1\ 0 \ldots 0]$ in which the unit value is in the ith column. For the matrix $(I - J)$ to be nonsingular, it must be the case that the largest eigenvalue of the Jacobian J is not equal to one; certainly, this is the case at any stable equilibrium.[44]

The direct effect on demand x_i from a change in quality q_j given by the matrix

$$D_q = \left[\frac{\partial x_m}{\partial q_n} \right] \tag{47}$$

[44] The requirement for a nonsingular matrix $(I - J$

From equation 37, the terms of D_q can be decomposed as

$$\frac{\partial x_i}{\partial q_j} = \begin{cases} -\ \ N P_i (1 - P_i) & \text{when } i = j \\ \ \ N P_i P_j & \text{otherwise} \end{cases} \tag{48}$$

From the effects 38 and 48, the total derivative of demand with respect to quality is given by

$$\frac{dx_i}{dq_j} = e_i \left(J^0 + J^1 + \ldots \right) D_q e_j^T \tag{49}$$

$$= e_i (I - J)^{-1} D_q e_j^T \tag{50}$$

Similarly, the derivative of demand with respect to compatibility involves both the direct impact of the compatibility on network sizes and the indirect effect as the change to network sizes propagate through the network term. The direct effect can be characterized by the matrix

$$D_{\varphi_k} = \left(\frac{\partial x_i}{\partial v_k} \frac{\partial v_k}{\partial z_k} \frac{\partial z_k}{\partial \rho_{k,j}} \frac{\partial \rho_{k,j}}{\partial \varphi_{k,j}} + \frac{\partial x_i}{\partial v_j} \frac{\partial v_j}{\partial z_j} \frac{\partial z_j}{\partial \rho_{j,k}} \frac{\partial \rho_{j,k}}{\partial \varphi_{k,j}} \right) \tag{51}$$

where the second term characterizes the complementary nature of compatibility when two-way adapters can be constructed.[45] Note that since products are considered to be fully compatible with themselves, for $i = j$.

It is worth noting that in the limit as the number of consumers becomes large, with full compatibility, the elasticity approaches that found in a market without network externalities. Intuitively, any loss in network size is made up by a corresponding gain in the network size of compatible products, resulting in no net impact to the effective network size.[47] Whether relative or absolute network size is considered, with full compatibility, as the number of consumers becomes large the model converges to the traditional MNL formulation.

C Existence, Uniqueness, and Stability

Miyao and Shapiro [14] establish existence, uniqueness, and stability for the general case of models of discrete choice. Their results are extended here to incorporate the Network MNL framework.

[47] More precisely, in equation 38, $\partial v_k / \partial x_j = 0$ for all j, k.

Given a set of prices p_i and product qualities q_i, we can consider consumer n's utility to be a function of consumer n's perceived network size of product i

$$u_{i,n} = q_i - \gamma p_i + v(z_{i,n}) + \sigma_n \tag{54}$$

where the network size of product i from the perspective of consumer n, $z_{i,n}$, is the consumer n's perception of the network size of product i; for example, if $z_{i,n}$ represents the sum of expected consumption of product i

In considering questions of existence, uniqueness and stability, the nature of equilibrium can be thought of in terms of a dynamic adjustment process. Taking other consumers' choices from time t as given and denoting the vector of consumption of product i by x^t, each consumer's choice probability of selecting product i at time $t + 1$ is given by

$$\Psi_{i,n}^{t+1}(x^t) = P\left(u_{i,n}^{t+1}(x^t) = \max_{j=1,\dots,I} u_{j,n}^{t+1}(x^t)\right) \tag{56}$$

and consumption of product i at time $t + 1$ is given by $x_{i,n}^{t+1} = N\Psi_{i,n}(x^t)$. In other words, in each time period, consumers simultaneously make a decision regarding choice probabilities based on the choice probabilities of their peers from the previous period.

Equilibrium is defined to be an allocation $x = [x_{1,1}, \dots, x_{I,N}]$ in which the expected number of consumers who select choice i is given by x_i where

$$x_{i,n} = \sum_{n=1}^{N} \Psi_{i,n}(x) \tag{57}$$

Choice probabilities are assumed to be symmetric; that is, $\Psi_{i,m} = \Psi_{i,n} \quad n, m$ $\{1, \ldots, N\}, m = n$ which implies $z_{i,m} = z_{i,n}$.

This allows us to write z_i in place of $z_{i,n}$, Ψ_i in place of $\Psi_{i,n}$, and equilibrium to be redefined as an allocation $x = [x_1, \ldots, x_I]$ in which the number of consumers who select choice i is given by x_i and $x_i = N\Psi_i(x)$.

A simple integration will show that a closed-form solution for Ψ is given by[48]

$$\Psi_i^{t+1}(x^t) = \frac{e^{q_i - \gamma p_i + v(z_i^t)}}{\sum_{j=1}^{I} e^{q_j - \gamma p_j + v(z_j^t)}} \tag{58}$$

From this it is easy to show that at least one equilibrium exists.

[48] See appendix A

C.1 Existence

Proposition C.1 (Existence) *An equilibrium exists.*

Proof C.1 *From equation 58,* $0 < \Psi_i(x) < 1 \quad i$ *and* $\square_i \Psi_i(x) = 1$*. And since* $v(z_i)$ *is continuous,* $\Psi_i(x)$ *is continuous for all* i*.*

Then $x_i = N\Psi_i(x)$ *is a function which maps the closed, convex ball* $B = \{x \ \mathbb{R}^I : 0 \leq x_1, \ldots, x_I \leq N\}$ *onto itself and by Brouwer's fixed point theorem, a fixed point* x *exists.* □

While proposition C.1 guarantees the existence of at least one equilibrium, a natural follow-up to the question of existence is that of the nature of equilibria.

C.2 Uniqueness

By virtue of the increasing returns nature of positive network externalities, multiple equilibria are to be expected and do, in fact, commonly arise. Con-

uct i at allocation x.

Proof C.2 *Define the mapping* Ω *as*

$$\Omega_i(x^t) = x_i^t - x_i^{t+1} \tag{60}$$

With Ψ *and* u *di erentiable,* Ω *is a function* $\Omega : B^l \rightarrow R^l$ *which maps the surface* $B = \{\omega \quad R^{l+} : \sum_{i=1}^{l} \omega_i = N\}$ *onto* R^l.

The Jacobian of Ω *is given by*

$$J = N[\alpha_{i,j}] = N(I + A) \tag{61}$$

where

$$A = [a_{i,j}] = -\left[\frac{\partial \Psi_i}{\partial u_j}\frac{\partial u_j}{\partial x_j}\right] \tag{62}$$

and, from equation 58,

$$\frac{\partial \Psi_i}{\partial u_j} = \begin{cases} \Psi_i(1-\Psi_i) & \textit{when } i = j \\ -\Psi_i\Psi_j & \textit{otherwise} \end{cases} \tag{63}$$

When network externalities are positive, $\frac{\partial \Psi_i}{\partial u_i}\frac{\partial u_i}{\partial x_i} > 0$ *and* $\frac{\partial \Psi_i}{\partial u_i}\frac{\partial u_i}{\partial x_j} < 0$*. From assumption 59, we know that* $\alpha_{i,i} = 1 - \frac{\partial \Psi_i}{\partial u_i}\frac{\partial u_i}{\partial x_i} > \frac{\partial \Psi_i}{\partial u_i}\frac{\partial u_i}{\partial x_i}$*. And since* $\sum_i \Psi_i = 1$*, we know that* $\frac{\partial \Psi_i}{\partial u_i}\frac{\partial u_i}{\partial x_i} + \sum_{j=i}\frac{\partial \Psi_i}{\partial u_j}\frac{\partial u_j}{\partial x_j} = 0$*. Hence,* $\alpha_{i,i} > -\sum_{j=i}\frac{\partial \Psi_i}{\partial u_j}\frac{\partial u_j}{\partial x_j} = \sum_{j=i}\alpha_{i,j}$.

Thus, because[49] J *is a dominant diagonal matrix with positive diagonal elements,* Ω *is univalent in* B^l *and the equilibrium* $\Omega(x\ ) = 0$ *is unique.* □

[49] *See [5, p. 84].*

When more than one equilibrium arises,[50] there tend to be a set of stable *where* Ψ_i *is the probability of any consumer* $k \in \{1, \ldots, K\}$ *purchasing product* i *at an equilibrium allocation* x^**.*

Proof C.3 *The dynamic adjustment process can be linearized around* x^* *with a matrix* Ω *such that* $x^{t+1} \approx x^* + \Omega(x^t - x^*)$ *where* $\Omega = [\omega_{i,j}]$ *and*

$$\omega_{i,j} = \frac{\partial \Psi_i}{\partial u_i} \frac{\partial u_i}{\partial v} \frac{\partial v}{\partial x_j}\bigg|_{x^*} \tag{65}$$

From [10][51], we know that magnitude of any eigenvalue λ *of* Ω *can be bounded by the following relationship:* $|\lambda| \leq \max_i \sum_j |\omega_{i,j}|$*. Since we know that* $\sum_i \Psi_i = 1$*, taking a derivative tells us that* $\frac{d\Psi_i}{du_i}\frac{du_i}{dv}\frac{dv}{dx_i} = -\sum_{j \neq i} \frac{d\Psi_j}{du_j}\frac{du_j}{dv}\frac{dv}{dx_j}$*. Thus we know that the equilibrium* x^*

$$
\begin{aligned}
|\lambda| &\le \max_i \sum_j |\omega_{i,j}| \\
&= \max_i \left(\frac{d\Psi_i}{du_i}\frac{du_i}{dv}\frac{dv}{dx_i} + \sum_{j = i} \frac{d\Psi_i}{du_i}\frac{du_i}{dv}\frac{dv}{dx_j} \right) \\
&\le 2\max_i \left(\frac{d\Psi_i}{du_i}\frac{du_i}{dv}\frac{dv}{dx_j} \right) \\
&< 1
\end{aligned}
\tag{66}
$$

which is true when $\frac{du_i}{dv}\frac{dv}{dx_j} < \frac{1}{2\frac{d\Psi_i}{du_i}} \quad \forall j$*. Since from equation 58,*

$$
\frac{\partial \Psi_i}{\partial u_j} = \begin{cases} \Psi_i(1-\Psi_i) & \text{when } i = j \\ -\Psi_i\Psi_j & \text{otherwise} \end{cases}
\tag{67}
$$

from equation 58, this condition can be restated as equation 64. □

While there is no eas

any given equilibria can be shown to be locally stable without having to evaluate the eigenvalues of the Jacobian.

A full characterization of invariant sets is elusive in all but the simplest cases of dynamical systems; however, numerical methods exist to identify and characterize equilibria and their accompanying manifolds.

With a single consumer type and 3 producers, equilibria can be defined as fixed points on a 3-dimensional simplex of choice probabilities. The system dynamics of consumer behavior can be considered to be determined by the tatônnement process described above.

D Local Uniqueness

Given that the increasing returns nature of the Network MNL gives rise to ing to know that equilibria are “locally isolated” or “locally unique” in the sense that there are no other arbitrarily close equilibria.

Formally,

Definition D.1 (Locally Unique) *Let $E(\omega)$ represent the set of equilibria demand vectors at a vector of demand parameters ω. An equilibrium Σ is locally unique if there exists neighborhoods U of ω and V of Σ such that for all ω*

in equations 46, 50, and 53 are nonsingular.

Proposition D.1 (Regularity) *Any equilibrium of the Network MNL at which price, quality, and compatibility derivatives are defined is also a regular equilibrium.*

Proof D.1

back into equilibrium.[52] *However, from [19],*[53] *we know that for any set of prices and product qualities a unique equilibrium exists. Furthermore, by [19] theorem 2.2, we know that demand is a continuous function of prices and product qualities. Thus, we know that Jacobians with respect to price and quality of the traditional multinomial logit model must be nonsingular.*

Likewise, if the network term is considered in terms of an additional product quality term of the traditional multinomial logit demand system, the Jacobian of direct e ect on demand of a change in product compatibility levels can be thought of as a linear combination of changes in the product quality term representing the network e ects or externalities. By similar reasoning, the Jacobian of the direct demand e ects of a change in product compatibility must be nonsingular as well. □

Finally, by the definition of regular equilibrium, we can say

Proposition D.2 (Local Uniqueness) *Every regular equilibrium is locally*

[52] *A technical description of this intuition is beyond the scope of this appendix. The interested reader is referred to [15] for more detail.*

[53] *See section 7.10.1.*

unique.

Proof D.2 *This follows as a direct consequence of the inverse function theorem.*[54]

which also tells us that any equilibrium of the Network MNL at which price, quality, and compatibility derivatives are defined is also locally unique.

D.1 Dominance

As is common with dynamical systems with positive feedback, equilibria in the Network MNL are prone to discontinuities in the set of stable equilibria. For example, while positive network externalities may be strong enough to support a stable equilibria dominant in each producer's product at a given set of prices, if a producer raises its price su□ciently, the network externalities attributable to a dominant network are no longer su□cient to overcome the price disadvantage.

[54] *See [1] appendix M.E.*

The profits enjoyed by a dominant producer in a market with strong positive network effects or externalities can be considerably greater than those seen by the fringe producers.[55]

Given a stable equilibria x , a dominant producer (say, producer i) may be interested in whether a change in price (say, by δ) would result in the loss of a stable, dominant equilibria in their good.

With the consumer response function Ψ_i given by[56]

$$\Psi_i(x; q, p, \gamma, \varphi) = \frac{e^{y+q_i-\gamma p_i+v_i(x)}}{\sum_{j=1}^{I} e^{y+q_j-\gamma p_j+v_j(x)}} \qquad (68)$$

an equilibrium is defined as any x such that $x - N\Psi(x; q, p, \gamma, \varphi) = 0$. Define the difference

Then the producer is interested in whether there is a solution of

$$F_i(x + \delta; q, p + \quad , \gamma, \varphi_{ii}$$

At the point of singularity, a bifurcation[57] will typically be observed.

E Model Formulation

Equilibria were identified and characterized by a 3-stage numerical simulation in the presence of both symmetric and asymmetric preferences. In the first stage, consumer behavior was described as a discrete dynamical system on the basis of the consumer tatônnement process and box coverings of chain recurrent sets[58] were found using a multilevel subdivision technique via GAIO.[59] In the second, equilibrium conditions were described as an NLP and the box coverings were used to populate the set of initial conditions. The NLP was solved in GAMS[60] to identify the equilibria of the system. In the

[57] [11] pp. 59–69 contains a brief overview of bifurcations.

[58] The chain recurrent sets can be considered the invariant sets of the system, where the definition of chain recurrence is useful for numerical simulations. See [3] for more information.

[59] GAIO (Global Analysis of Invariant Objects), created by Michael Dellnitz and Oliver Junge, consists of a C library and Matlab or Python interfaces which can identify and characterize attributes of dynamical systems, such as equilibria, stable manifolds and other invariant sets. For more information on GAIO, see [2]

[60] GAMS (General Algebraic Modeling System) is a modeling system created by Alexander Meeraus which allows an optimization problem to be described algebraically and solved with any of a variety of solvers. For more information on GAMS, see [12].

From equation 3, we solve

$$\min_{x} \square \square \square \square_{i} \square x$$

tion, we do not need to specify utility maximization complementary slackness

$$\frac{dx_i}{dp_{ii}} = \sum_{iii} a_{i,iii} \frac{\partial x_{iii}}{\partial p_{ii}} \tag{75}$$

E.2.2 Profit Maximization in Compatibility

$$N(p_i - b_i)\frac{dx_i}{d\varphi_{i,ii}} - \frac{dc}{d\varphi_{i,ii}} \leq 0 \qquad \varphi_{i,ii} \geq 0 \tag{76}$$

where

$$\frac{dx_i}{d\varphi_{i,ii}}\text{d}$$

E.2.4 Supplemental Equations

$$v_i = (z_i/Z)^2 \tag{79}$$

$$z_i = (N-1)\left(x_i + \sum_{ii=i} \rho_{i,ii} x_i\right) \tag{80}$$

$$\frac{\partial x_i}{\partial v_{ii}} = \begin{cases} x_i(1-x_i) & \text{when } i = ii \\ -\ x_i x_{ii} & \text{otherwise} \end{cases} \tag{81}$$

$$\frac{\partial x_i}{\partial p_{ii}} = \begin{cases} -\ \gamma x_i(1-x_i) & \text{when } i = ii \\ \gamma x_i x_{ii} & \text{otherwise} \end{cases} \tag{82}$$

$$\frac{dv_i}{dx_i} = (N-1)\frac{\partial v_i}{\partial z_i}\rho_{i,ii} \tag{83}$$

$$\rho_{i,ii} = \begin{cases} 1 & \text{when } i = ii \\ \varphi_{i,ii}/(1 + \varphi_{i,ii}) & \text{otherwise} \end{cases} \tag{84}$$

$$\frac{\partial v_i}{\partial z_i} = 2z_i/Z^2 \tag{85}$$

$$\frac{\partial z_i}{\partial \rho_{i,ii}} = (N - 1)x_i \tag{86}$$

$$\frac{\partial \rho_{i,ii}}{\partial \varphi_{i,ii}} = \begin{cases} 0 & \text{when } i = ii \\ 1/(1 + \varphi_{i,ii})^2 & \text{otherwise} \end{cases} \tag{87}$$

$$\frac{\partial x_i}{\partial \varphi_{ii,iii}} = \begin{cases} 0 & \text{when } ii = iii \\ \frac{\partial x_i}{\partial v_{ii}} \frac{\partial v_{ii}}{\partial z_{ii}} \frac{\partial z_{ii}}{\partial \rho_{ii,iii}} \frac{\partial \rho_{ii,iii}}{\partial \varphi_{ii,iii}} & \text{otherwise} \end{cases} \tag{88}$$

Note that equation 88 reflects a one-way adapter. A two-way adapter would be reflected by an additional term resulting in entry i, ii, iii being given by an expression such as $\frac{\partial x_i}{\partial v_{ii}} \frac{\partial v_{ii}}{\partial z_{ii}} \frac{\partial z_{ii}}{\partial \rho_{ii,iii}} \frac{\partial \rho_{ii,iii}}{\partial \varphi_{ii,iii}} + \frac{\partial x_i}{\partial v_{iii}} \frac{\partial v_{iii}}{\partial z_{iii}} \frac{\partial z_{iii}}{\partial \rho_{iii,ii}} \frac{\partial \rho_{iii,ii}}{\partial \varphi_{ii,iii}}$ where $ii = iii$.

$$\frac{dc}{d\varphi}$$

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