Graph Neural Networks for Causal Inference Under Network Confounding

M c ae P Leun † Pante s Loupos‡

Marc

Abstract. [illegible]

1 Introduction

U VA

U VA

x

x **m** **y**

$\boldsymbol{X}$

$\boldsymbol{A}$ A

1.2 Related Literature

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A i $n(i, 1$

2 Setup

$\mathcal{N}_n$ $1, \ldots, n$ $\boldsymbol{A}$
i $\mathcal{N}_n$ $(\varepsilon_i, \nu_i$ $\mathbb{R}^{d_\varepsilon}$ $\mathbb{R}^{d_\nu}$ X_i $\mathbb{R}^{d_x}$

$$Y_i \quad g_n(i, D, X, A, \varepsilon \qquad D_i \quad h_n(i, X, A, \nu \ ,$$

$\boldsymbol{X}$ $(X_i \ {}_{i=1}^{n}$ i X_i' $\boldsymbol{Y}$ $\boldsymbol{D}$ $\boldsymbol{\varepsilon}$
$\boldsymbol{\nu}$ $(g_n, h_n \ {}_{n \in \mathbb{N}}$
$g_n($ $\mathbb{R}$ $h_n($ $0, 1$
$(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{X}, \boldsymbol{A}$ $(\boldsymbol{A}, \boldsymbol{X}, \boldsymbol{\varepsilon}, \boldsymbol{\nu}$
$(\boldsymbol{X}, \boldsymbol{A}$

$(\boldsymbol{A}, \boldsymbol{X}, \boldsymbol{\varepsilon}, \boldsymbol{\nu}$
$h_n($ $g_n($
$g_n($
$h_n($ Y_i D_i

E p 1

$$Y_i \quad \alpha + \beta \frac{\sum_{j=1}^{n} A_{ij} Y_j}{\sum_{j=1}^{n} A_{ij}} + \frac{\sum_{j=1}^{n} A_{ij} Z_j'}{\sum_{j=1}^{n} A_{ij}} \gamma + Z_j' \delta + \varepsilon_i,$$

Z_i $(D_i, X_i'\ '$ β

1 n

A

$$Y \quad \frac{\alpha}{1 \quad \beta} 1 + Z\delta + \gamma\beta \sum_{k=0} \beta^k \tilde{A}^{k+1} Z + \sum_{k=0} \beta^k \tilde{A}^k \varepsilon.$$

$$Y_i \quad g_n(i, D, X, A, \varepsilon$$

E p

$$D_i \quad 1 \quad \alpha + \beta \frac{\sum_{j=1}^n A_{ij} D_j}{\sum_{j=1}^n A_{ij}} + \frac{\sum_{j=1}^n A_{ij} Z_j'}{\sum_{j=1}^n A_{ij}} \gamma + Z_i'\delta + \nu_i \quad 0 \; .$$

U

$(X, A, \nu$

D D_i $h_n(i, X, A, \nu$

i $(\nu_i, X, A$

D_j $\sigma_j(X, A$

$D_j \quad 1$ D_i $h_n(i, X, A, \nu_i$

E p

D_i i Y_i

$$Y_i \quad g_n(D_{\mathcal{N}(i,K)}, \varepsilon_i \qquad D_i \quad 1 \; W_i'\gamma \quad \nu_i \; ,$$

W_i $(X, A$ K

$$Y_i(d \quad g_n(i, d, X, A, \varepsilon \; .$$

$Y_i(d$ D_i

$(\boldsymbol{X}, \boldsymbol{A}$

ε $\boldsymbol{\nu}$

Assumption 1 U o n $\mathbf{n}$ $\mathbb{N}$, ε $\boldsymbol{\nu}$ $\boldsymbol{X}, \boldsymbol{A}$

A

$(\boldsymbol{X}, \boldsymbol{A}$

U VA $\boldsymbol{\varepsilon}_{\mathbf{i}}$

$\boldsymbol{\nu}_{\mathbf{i}}$ $\mathbf{X}_{\mathbf{i}}$

$\mathbf{S}$ $\mathcal{N}_{\mathbf{n}}$ $\boldsymbol{D}_{\mathbf{S}}$ $(\mathbf{D}_{\mathbf{i}}$ $_{\mathbf{i} \in \mathbf{S}}$ $\boldsymbol{X}_{\mathbf{S}}$

$\boldsymbol{A}_{\mathbf{S}}$

s

s $\gamma_n(s$

s Y_i

i i

i

D_i

E p

$\sup_n \gamma_n(s$ $C|\beta|^s$

C 0

$\sup_n \eta_n(s$ s

Y_i

D $D_{\mathcal{N}(i,K)}$ $\gamma_n(s$ $c\,1$ s K

c W_i X_i

$\eta_n(s$ 0 s

2.1 Related Literature

U VA

$$Y_i \quad g(D_i, X_i, \varepsilon_i \qquad \varepsilon_i \quad D_i \quad X_i.$$

$$T_i \quad f_n(i, D, A \qquad W_i \quad q_n(i, X, A$$

$f_n($ $q_n($ n

o u n A

$Y_i = g(T_i, W_i, \varepsilon_i)$ ε_i T_i W_i,

T_i W_i

T_i W_i

$$\tilde{T}_i = \left(D_i, \sum_{j=1}^{n} A_{ij} D_j\right) \qquad \tilde{W}_i = \left(X_i, \sum_{j=1}^{n} A_{ij}, \frac{\sum_{j=1}^{n} A_{ij} X_j}{\sum_{j=1}^{n} A_{ij}}\right).$$

T_i Y_i D

$D_{\mathcal{N}(i,1)}$

W_i

K

$D_{\mathcal{N}(i,K)}$ K

T_i

W_i

W_i X_i

$\boldsymbol{X}$ $\boldsymbol{A}$

$(\boldsymbol{X}, \boldsymbol{A}$

$\mathbf{t}$ $(1, 0$ $\mathbf{t}'$ $(0, 0$

$\mathcal{M}_{\mathbf{n}}$

$\mathbf{n}(\mathbf{i}, 1$ $|\mathcal{N}(\mathbf{i}, 1\ |$

$\mathbb{R}_{+}$ 0

$\mathcal{M}_{\mathbf{n}}$ $\mathbf{i}$

$($ $+$ $($

$(X, A, \nu$ $\mathbf{T_i}$ $\mathbf{Y_i}$

D

$\tau(t, t'$

U $\tau(t, t'$

$\tau(t, t'$

ν_i

i D_i $h_n(i, X, A, \nu_i$

$(X, A$

V

$\tau(t, t'$

A A A

$$\hat{\tau}_i(t, t' \qquad \frac{1 \ T_i \quad t \ (Y_i \quad \hat{\mu}_t(i, X, A}{\hat{p}_t(i, X, A} + \hat{\mu}_t(i, X, A$$

$$\frac{1 \ T_i \quad t' \ (Y_i \quad \hat{\mu}_{t'}(i, X, A}{\hat{p}_{t'}(i, X, A} \quad \hat{\mu}_{t'}(i, X, A \ .$$

3.1 Architecture

$\mathbf{X}_i$

$\Gamma($ $\boldsymbol{\mu}($, $\boldsymbol{\sigma}($, $\Sigma($, $\min($, $\max($

$\Gamma($ $\Gamma_1($

$$\Gamma_1(\quad \boldsymbol{\mu}(\quad \boldsymbol{\sigma}(\quad \Sigma(\quad \min(\quad \max($$

$\mathbf{n}(\mathbf{i}, 1$

$\Gamma_1($

$| \, |$

$$\mathbf{S}(\, , \boldsymbol{\alpha} \quad \left(\frac{\log(| \, | + 1}{\boldsymbol{\delta}} \right)^{\boldsymbol{\alpha}}, \quad \boldsymbol{\delta} \quad \frac{1}{\mathbf{n}} \sum_{\mathbf{i}=1} \log \left(\sum_{\mathbf{j}=1} \mathbf{A}_{\mathbf{ij}} + 1 \right), \quad \boldsymbol{\alpha} \quad 1, 1 \, .$$

$\boldsymbol{\alpha}$ $\boldsymbol{\alpha}$ 1 $\boldsymbol{\alpha}$

1 $\boldsymbol{\alpha}$ 0

$$\hat{\mathbf{f}}_{\mathrm{GNN}} \quad \underset{\mathbf{f} \in \mathcal{F}_{\mathrm{GNN}}(\mathbf{L})}{\operatorname{argmin}} \quad {}_{\mathbf{i} \in} \quad \ddot{\mathbf{y}}$$

A **f** $\mathcal{F}_{\mathrm{GNN}}($**L** i i

i $(\pi_i(X, \pi_i(A$

ropo on 1 *u o o ny* **n** $\mathbb{N}$ *n u on* π,

$$f_n(i, D, A \quad f_n(\pi(i, \pi(D, \pi(A,$$

$$g_n(i, D, X, A, \varepsilon \quad g_n(\pi(i, \pi(D, \pi(X, \pi(A, \pi(\varepsilon, \quad n$$

$$\phi_{t,t'}(i \quad \frac{1 \;\; T_i \quad t \;(Y_i \quad \mu_t(i, X, A}{p_t(i, X, A} + \mu_t(i, X, A$$

$$\frac{1 \;\; T_i \quad t' \;(Y_i \quad \mu_{t'}(i, X, A}{p_{t'}(i, X, A} \quad \mu_{t'}(i, X, A \quad \tau(t, t' \;,$$

$i \quad \mathcal{M}_n$

$$\sigma_n^2 \quad \mathrm{V} \quad \frac{1}{\overline{m_n}} \sum_{i \in \mathcal{M}_n} \phi_{t,t'}(i \quad X, A \quad .$$

Assumption $\quad M \quad n \quad p \quad 4 \quad u \quad o \quad ny$
$n \quad \mathbb{N}, \; i \quad \mathcal{N}_n, \quad n \quad d \quad 0, 1^{\,n}, \quad ||Y_i(d|^p \quad X, A \quad M$
$|\underline{\pi}, \pi \quad (0, 1 \quad u \quad \hat{p}_t(i, X, A, p_t(i, X, A \quad |\underline{\pi}, \pi \quad n$

$\phi_{t,t'}(i \quad {}_{i=1}^{n} \quad \psi$

$\phi_{t,t^1}($i $_{i=1}^{n}$ ψ

$\Lambda_n($s 2M M

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A

A

or n $\phi_{t,t'}(i$ n $\tau(t,t'$ n n on o $\phi_{t,t'}(i$ $\tau_i(t,t'$
$|Y_i$ T_i $t, X,$

$\hat{\sigma}^2$

A

5 Approximate Sparsity

A L

$(X_{\mathcal{N}(i,L)})$

$(W_i$ $_{i=1}^{n}$ ν $(\nu_i$ $_{i=1}^{n}$

$$V_i(W, \nu; \theta \quad \alpha + \beta \frac{\sum_{j=1}^{n} A_{ij} W_j}{\sum_{j=1}^{n} A_{ij}} + \delta \frac{\sum_{j=1}^{n} A_{ij} X_j}{\sum_{j=1}^{n} A_{ij}} + \gamma X_j + \nu_i + \frac{\sum_{j=1}^{n} A_{ij} \nu_j}{\sum_{j=1}^{n} A_{ij}}$$

θ $(\alpha, \beta, \delta, \gamma$ Y_i $_{i=1}^{n}$

Y_i $V_i(Y, \varepsilon; \theta_y$ θ_y $(0.5, 0.8, 10,$ 1 D_i $_{i=1}^{n}$

D_i 1 $V_i(D, \nu; \theta_d$ 0 θ_d $($ $0.5, 1.5, 1,$ 1

D_i^0 $_{i=1}^{n}$ D_i^0 1 $V_i(0, \nu; \theta_d$ 0

$\nu_i + \sum_{j=1}^{n} A_{ij} \nu_j$ $\sum_{j=1}^{n} A_{ij}$

A A

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A

n 2 $|Y_i$ T_i t, X, A

T_i t n $1000, 2000, 4000$

6.2 Nonparametric Estimators

A $\Gamma_2($

L $1,$

L 1

	$L = 1$			$L = 2$			$L = 3$		
n H	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\hat{\tau}(1,0)$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
C	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
C	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$W\ \hat{\tau}(1,0)$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
W C	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
W	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
C	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

si u ations, he esti and is $\boldsymbol{\tau}(1,0) = 0$, treated $\approx$

A **L** 2

L $2, 4, 8, \ldots$

L 2

L

7.1 Comparison with He and Song (2024)

$\tau(t, t'$

A A

A

$\mathbf{n}$ 4413

[0.01, 0.99

A

$\mathbf{T_i}^{(1)}$

	ADM	GNN			GLM		
		Layer	Layer	Layer	Order	Order	Order
Leader case							
$\mathbf{G}_{ee}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{sc}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{all}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Leader adopter case							
$\mathbf{G}_{ee}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{sc}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{all}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Adopter case							
$\mathbf{G}_{ee}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{sc}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{all}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

$\mathbf{n} = 4413$

$\mathbf{T_i}^{(1)}$

L

A

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A

$\mathbf{T_i}^{(2)}$ $2 +$

$\mathbf{T_i}^{(1)}$

A

A

$\mathbf{T_i}^{(1)}$

A Additional Results on GNNs

A

$$\frac{1}{m_n}\sum_{i\in\mathcal{M}_n}\left(\hat{p}_t(i,X,A) \quad p_t(i,X,A)\right)^2 \quad o_p(n^{-1/2}).$$

U

$$\frac{1}{m_n}\sum_{i\in\mathcal{M}_n}\left(\hat{p}_t(i,X,A) \quad p_t(i,X_{\mathcal{N}(i,L)},A_{\mathcal{N}(i,L)})\right)^2 \quad o_p(n^{-1/2}). \qquad \text{A}$$

$\hat{p}_t(i,X,A)$ L $(X_{\mathcal{N}(i,L)},A_{\mathcal{N}(i,L)})$

$p_t(i,X_{\mathcal{N}(i,L)},A_{\mathcal{N}(i,L)})$

A

A

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{p}_t(i,X,A) \quad p_t(i,X_{\mathcal{N}(i,L)},A_{\mathcal{N}(i,L)})\right)^2 \quad C\left(\frac{WL\log R}{n}\log n + \frac{\log\log n + \gamma}{n}\right) + \epsilon^2$$

A

$1 \quad e^{-\gamma}$ W C

$n \quad R$

ψ

A A

W, R, L, n A

A.1 WL Function Class

n n

$(\boldsymbol{X}, \boldsymbol{A}$, $(\boldsymbol{X}', \boldsymbol{A}'$ **f** $\mathbf{f}(\boldsymbol{X}, \boldsymbol{A} \neq \mathbf{f}(\boldsymbol{X}', \boldsymbol{A}'$

W n W

$(\boldsymbol{X}, \boldsymbol{A}$

$\mathbf{H}^2$

$(\mathbf{h}, \mathbf{h}'$ $\boldsymbol{\rho}(\mathcal{F}$ $\mathbf{f}(\mathbf{h}$ $\mathbf{f}(\mathbf{h}'$ $\mathbf{f}$ $\mathcal{F}$.

$\mathcal{E}, \mathcal{F}$ $\mathbf{H}$ $\mathcal{E}$ o

n $\mathcal{F}$ $\boldsymbol{\rho}(\mathcal{F}$ $\boldsymbol{\rho}(\mathcal{E}$

A $\mathcal{E}$

$\mathcal{F}$

$\mathcal{F}$ $\mathbf{H}$ $\mathcal{E}$

$\mathbf{f}$ $_{\mathrm{L},\mathbf{L}}$ $(\boldsymbol{X}, \boldsymbol{A}$ $\Sigma^{\mathbf{n}}$

$\mathbf{L}$ $\mathcal{C}(\mathbf{H}$

$\mathbf{H}$ $\mathbf{L}$ $\mathbb{N}$ W un on

$$\mathcal{F}_{\mathrm{L}}(\mathbf{L} \quad \mathbf{f} \quad \mathcal{C}(\mathbf{H} : \boldsymbol{\rho}(\mathbf{f}_{\mathrm{L},\mathbf{L}} \quad \boldsymbol{\rho}(\mathbf{f} \quad .$$

$(\boldsymbol{X}, \boldsymbol{A}$

$\mathbf{L}$

$\mathbf{p_t}($ $\boldsymbol{\mu}_\mathbf{t}($

$\mathcal{F}$ $_{\mathrm{L}}$(**L** **L**

$\mathcal{F}_{\mathrm{GNN}}$ (**L**

A

A

T L

A

f $\mathcal{F}_{\text{GNN}}$ (L

$$(X, A \quad f(i, X, A), \ldots, f(i, X, A) \quad \mathbb{R}^n$$

n ti es

L +1

$\varphi_{L+1}(h_i^{(L)}, h_j^{(L)} : j \quad \mathcal{N}_n$

$\mathcal{F}_{\text{GNN}}$ (L A

$X_i \quad \rho(\mathcal{F}_{\text{GNN}}$ (L $\rho($ f $_{L,L}$ L

L ■

A.2 Disadvantages of Depth

L A

L L

r turn to p t A

L

U

n 1 n 1

B Verifying §8 Assumptions

A A

A

max $\mathbf{y}_{\mathbf{n}}(\mathbf{s}$ 2 , $\boldsymbol{\psi}_{\mathbf{n}}(\mathbf{s}$

($\mathbf{c}(1$ 4 $\mathbf{p}$ $^{-1}\mathbf{s}$ $\mathbf{c}$ 0 $\mathbf{p}$ A A

$$\sup_{\mathbf{n}} \max_{\mathbf{i} \in \mathcal{N}_n} |\mathcal{N}_{\boldsymbol{A}}(\mathbf{i}, \mathbf{s}| \quad \mathbf{C}\mathbf{s}^{\mathbf{d}}$$

$\mathbf{C}$ 0 $\mathbf{d} \succ 1$

$D'_{\mathbf{B}}$ $(\mathbf{D}'_{\mathbf{j}})_{\mathbf{j}\in\mathbf{B}}$ $\mathbf{B}$ $\mathcal{N}_{\mathbf{n}}$ U /

$$\mathbf{p_t}(\mathbf{i}, X, A) \quad (\mathbf{D}'_{\mathbf{i}} + (\mathbf{D}_{\mathbf{i}} \quad \mathbf{D}'_{\mathbf{i}}) \,|\, \mathbf{a}, \mathbf{b}, \; \mathbf{V}'_{\mathbf{i}} + (\mathbf{V}_{\mathbf{i}} \quad \mathbf{V}'_{\mathbf{i}}) \,|\, \boldsymbol{\alpha}, \boldsymbol{\beta} \quad X, A)$$
$$(\mathbf{D}'_{\mathbf{i}} \,|\, \mathbf{a} \quad , \mathbf{b} + \; , \; \mathbf{V}'_{\mathbf{i}} \,|\, \boldsymbol{\alpha} \quad , \boldsymbol{\beta} + \quad X, A)$$
$$+ \mathbf{I} \; (|\mathbf{D}_{\mathbf{i}} \quad \mathbf{D}'_{\mathbf{i}}| \quad X, A) \; \mathbf{m} \; + \; (|\mathbf{V}_{\mathbf{i}} \quad \mathbf{V}'_{\mathbf{i}}| \quad X, A) \; \mathbf{n} \,.$$
$$\mathbf{R}_0$$

/

$(\mathbf{D}'_{\mathbf{i}} \,|\, \mathbf{a}, \mathbf{b}$

$$|p_t(i, X, A \quad p_t(i, X_{\mathcal{N}(i, r_\lambda(s+1))}, A_{\mathcal{N}(i, r_\lambda(s+1))}) | \quad \lambda_n(s \; +1 \quad +2R_0.$$

roo o

$$\mu_t(i, X, A \quad |Y_i 1_i(t \quad X, A \;\; p_t(i, X, A$$

$$B \quad \mathcal{N}(i, s \qquad Y_i' \quad g_{n(i,s)}(i, D_B', X_B, A_B, \varepsilon_B$$

$$| \quad |Y_i 1_i(t \quad X, A \qquad |Y_i' 1_i(t \quad X, A | \quad \gamma_n(s \quad +\Lambda_n(i, s \; n(i, s \; \eta_n(s$$

A 4

$|\mathbf{R}_1|$ $\boldsymbol{\lambda}_n$

$|R_1| \quad y_n(s \quad + \Lambda_n(i, s \; n(i, s \; \eta_n(s$ ■

C n Y_i', D_i' n n $1_i(t' \quad 1 \; D_i' \quad d, \; \check{r}^{n}_{j=1} A_{ij} D_j'$

n A on , , , n , C 0 o n

$n \quad \mathbb{N}, \; i \quad \mathcal{N}_n, \quad n \quad s \succ 0,$

$$|Y_i| 1_i(t \quad 1_i(t'| \quad X, A \quad C\,(1 \; + n(i, 1 \quad \eta_n(s\,.$$

Proof. a, b, α, β, $V_i \quad \check{r}^{n}_{j=1} A_{ij} D_j$ $V_i' \quad \check{r}^{n}_{j=1} A_{ij} D_j'$ $\mathcal{C} \quad |D_i \quad D_i'| \quad , |V_i \quad V_i'|$

$$|Y_i| 1_i(t \quad 1_i(t'| \quad X \quad x, A \quad a$$
$$|Y_i| 1_i(t \quad 1_i(t'| \quad \mathcal{C}, X \quad x, A \quad a \; + C \quad (\mathcal{C}^c \quad X \quad x, A \quad a$$

C 0 A A

$$1_i(t \quad 1 \quad D_i \quad |a, b\,, \; V_i \quad |\alpha, \beta \qquad 1_i(t' \quad 1 \quad D_i' \quad |a, b\,, \; V_i' \quad |\alpha, \beta \quad .$$

U $\mathcal{C}$

$$1 \; D_i \quad |a, b\,, \; V_i \quad |\alpha, \beta \; (\quad 1 \; D_i' \; + (D_i \quad D_i' \quad |a, b\,, \; V_i' \; + (V_i \quad V_i' \quad |\alpha, \beta$$
$$1 \; D_i' \quad |a \quad , b \; + \; , \; V_i' \quad |\alpha \quad s$$

$\mathbb{R}^{\mathbf{d}}$ $(\mathbf{f}$ $\mathbf{f}$ $\mathcal{L}_{\mathbf{d}}$

$$\mathcal{P}_{\mathbf{n}}(\mathbf{h},\mathbf{h}';\mathbf{s} \quad (\mathbf{H},\mathbf{H}' : \mathbf{H},\mathbf{H}' \quad \mathcal{N}_{\mathbf{n}}, \mathbf{H} \quad \mathbf{h}, \mathbf{H}' \quad \mathbf{h}', \square_{A}(\mathbf{H},\mathbf{H}' \succ \mathbf{s} \ .$$

D n on C 1 A $\mathbf{Z}_{\mathbf{i}}$ $^{\mathbf{n}}_{\mathbf{i}=1}$ *on on* $\boldsymbol{\psi}$ *n n n* $\mathcal{F}_{\mathbf{n}}$ $\mathbf{C}$ $(0,$ $\mathcal{F}_{\mathbf{n}}$ $\boldsymbol{\psi}_{\mathbf{n}}(\mathbf{s}$ $_{\mathbf{s},\mathbf{n}\in\mathbb{N}}$ $\boldsymbol{\psi}_{\mathbf{n}}(0$ 1 $\mathbf{n}$

$$| \quad (\mathbf{f}(Z_{\mathbf{H}} \ , \mathbf{f}'(Z_{\mathbf{H}'} \quad | \quad \mathbf{C}\mathbf{h}\mathbf{h}'(\|\mathbf{f}\|_{\infty} + \quad (\mathbf{f} \quad (\|\mathbf{f}'\|_{\infty} + \quad (\mathbf{f}' \quad \boldsymbol{\psi}_{\mathbf{n}}(\mathbf{s}$$

$\mathbf{n},\mathbf{h},\mathbf{h}'$ $\mathbb{N}$ $\mathbf{s}$ 0 $\mathbf{f}$ $\mathcal{L}_{\mathbf{h}}$ $\mathbf{f}'$ $\mathcal{L}_{\mathbf{h}'}$ $(\mathbf{H},\mathbf{H}'$ $\mathcal{P}_{\mathbf{n}}(\mathbf{h},\mathbf{h}';\mathbf{s}$ $\boldsymbol{\psi}_{\mathbf{n}}(\mathbf{s}$ *n n o n* $\mathbf{Z}_{\mathbf{i}}$ $^{\mathbf{n}}_{\mathbf{i}=1}$

C *n* A *u on , , , n , n n , o n* $\mathbf{t},\mathbf{t}'$ $\mathcal{T}$, $\boldsymbol{\phi}_{\mathbf{t},\mathbf{t}'}(\mathbf{i}$ $^{\mathbf{n}}_{\mathbf{i}=1}$ *on on* $\boldsymbol{\psi}$ *n n n* $(\boldsymbol{X},\boldsymbol{A}$ *n on n n o n* $\boldsymbol{\psi}_{\mathbf{n}}(\mathbf{s}$ *n n*

Proof. $\mathcal{F}_{\mathbf{n}}$ $\boldsymbol{\sigma}$ $(\boldsymbol{X},\boldsymbol{A}$ $(\mathbf{h},\mathbf{h}'$ $\mathbb{N}$ $\mathbb{N}$ $(\mathbf{f},\mathbf{f}'$ $\mathcal{L}_{\mathbf{h}}$ $\mathcal{L}_{\mathbf{h}'}$ $\mathbf{s}$ 0 $(\mathbf{H},\mathbf{H}'$ $\mathcal{P}_{\mathbf{n}}(\mathbf{h},\mathbf{h}';\mathbf{s}$ $\mathbf{Z}_{\mathbf{i}}$ $\boldsymbol{\phi}_{\mathbf{t},\mathbf{t}'}(\mathbf{i}$ $Z_{\mathbf{H}}$ $(\mathbf{Z}_{\mathbf{i}}$ $_{\mathbf{i}\in\mathbf{H}}$ $\boldsymbol{\xi}$ $\mathbf{f}(Z_{\mathbf{H}}$ $\boldsymbol{\zeta}$ $\mathbf{f}'(Z_{\mathbf{H}'}$

$$\mathbf{D}_{\mathbf{i}}^{(\mathbf{s})} \quad \mathbf{h}_{\mathbf{n}(\mathbf{i},\mathbf{s})}(\mathbf{i}, X_{\mathcal{N}(\mathbf{i},\mathbf{s})}, A_{\mathcal{N}(\mathbf{i},\mathbf{s})}, \nu_{\mathcal{N}(\mathbf{i},\mathbf{s})} \ .$$

$D_{\mathcal{N}(\mathbf{i},\mathbf{s}')}^{(\mathbf{s})}$ $(\mathbf{D}_{\mathbf{j}}^{(\mathbf{s})}$ $_{\mathbf{j}\in\mathcal{N}(\mathbf{i},\mathbf{s}')}$

$$1_{\mathbf{i}}^{(\mathbf{s})}(\mathbf{t} \quad 1 \ \mathbf{f}_{\mathbf{n}(\mathbf{i},\mathbf{s}/2)}(\mathbf{i}, D_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)}^{(\mathbf{s}/2)}, A_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)} \quad \mathbf{t} \ ,$$
$$\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})} \quad \mathbf{g}_{\mathbf{n}(\mathbf{i},\mathbf{s}/2)}(\mathbf{i}, D_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)}^{(\mathbf{s}/2)}, X_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)}, A_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)}, \varepsilon_{\mathcal{N}(\mathbf{i},\mathbf{s}/2)} \ ,$$
$$\mathbf{Z}_{\mathbf{i}}^{(\mathbf{s})} \quad 1_{\mathbf{i}}^{(\mathbf{s})}(\mathbf{t} \ (\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})} \quad \boldsymbol{\mu}_{\mathbf{t}}(\mathbf{i}, X, A$$

A $(\mathbf{Z}_i^{(s/2,\xi)}$ $_{i\in H}$ $(\mathbf{Z}_j^{(s/2,\zeta)}$ $_{j\in H^1}$ $\mathcal{F}_n$

$$| \quad (\boldsymbol{\xi}, \boldsymbol{\zeta} \quad \mathcal{F}_n | \quad | \quad (\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(s/2)}, \boldsymbol{\zeta} \quad \mathcal{F}_n | + | \quad (\boldsymbol{\xi}^{(s/2)}, \boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(s/2)} \quad \mathcal{F}_n |$$

$$2\|\mathbf{f}'\|_\infty \quad |\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(s/2)}| \quad \mathcal{F}_n \quad + 2\|\mathbf{f}\|_\infty \quad |\boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(s/2)}| \quad \mathcal{F}_n$$

$$2 \quad \mathbf{h}\|\mathbf{f}'\|_\infty \quad (\mathbf{f} \quad + \mathbf{h}'\|\mathbf{f}\|_\infty \quad (\mathbf{f}' \quad \max_{i \in \mathcal{N}}$$

$\mathcal{L}_{\mathbf{h}}$ $\mathcal{L}_{\mathbf{h}^1}$ $\mathbf{s}$ 0 $(\mathbf{H},\mathbf{H}'$ $\mathcal{P}_{\mathbf{n}}(\mathbf{h},\mathbf{h}';\mathbf{s}$

$$\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})} \quad \mathbf{g}_{\mathbf{n}(\mathbf{i},\mathbf{s})}(\mathbf{i}, D_{\mathcal{N}(\mathbf{i},\mathbf{s})}, X_{\mathcal{N}(\mathbf{i},\mathbf{s})}, A_{\mathcal{N}(\mathbf{i},\mathbf{s})}, \varepsilon_{\mathcal{N}(\mathbf{i},\mathbf{s})} \ ,$$

$\boldsymbol{\xi}$ $\mathbf{f}((\mathbf{Y}_{\mathbf{i}}\ _{\mathbf{i}\in\mathbf{H}}$ $\boldsymbol{\zeta}$ $\mathbf{f}'((\mathbf{Y}_{\mathbf{i}}\ _{\mathbf{i}\in\mathbf{H}^1}$ $\boldsymbol{\xi}^{(\mathbf{s})}$ $\mathbf{f}((\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})}\ _{\mathbf{i}\in\mathbf{H}}$ $\boldsymbol{\zeta}^{(\mathbf{s})}$ $\mathbf{f}'((\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})}\ _{\mathbf{i}\in\mathbf{H}^1}$
A

$$\begin{aligned} | \quad (\boldsymbol{\xi},\boldsymbol{\zeta} \ \ \mathcal{F}'_{\mathbf{n}} | \quad & | \quad (\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(\mathbf{s}/2)}, \boldsymbol{\zeta} \ \ \mathcal{F}'_{\mathbf{n}} | + | \quad (\boldsymbol{\xi}^{(\mathbf{s}/2)}, \boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(\mathbf{s}/2)} \ \ \mathcal{F}'_{\mathbf{n}} | \\ & 2\|\mathbf{f}'\|_{\infty} \ \ |\,|\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(\mathbf{s}/2)}| \ \ \mathcal{F}'_{\mathbf{n}} \ + 2\|\mathbf{f}\|_{\infty} \ \ |\,|\boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(\mathbf{s}/2)}| \ \ \mathcal{F}'_{\mathbf{n}} \\ & 2 \ \mathbf{h}\|\mathbf{f}'\|_{\infty} \quad (\mathbf{f} \quad + \mathbf{h}'\|\mathbf{f}\|_{\infty} \quad (\mathbf{f}' \ \max_{\mathbf{i}\in\mathcal{N}_n} \ |\,|\mathbf{Y}_{\mathbf{i}} \quad \mathbf{Y}_{\mathbf{i}}^{(\mathbf{s}/2)}| \ \ \mathcal{F}'_{\mathbf{n}} \\ & 2 \ \mathbf{h}\|\mathbf{f}'\|_{\infty} \quad (\mathbf{f} \quad + \mathbf{h}'\|\mathbf{f}\|_{\infty} \quad (\mathbf{f}' \ \ \boldsymbol{\gamma}_{\mathbf{n}}(\mathbf{s} \ \ 2 \ , \end{aligned}$$

A $\boldsymbol{\psi}$

A

$\hat{p}_t(i, X, A$ ... C, C' 0 R^2_{1t}

$\frac{1}{m_n} \sum_{i\in\mathcal{M}_n} \sum_{j\in\mathcal{M}_n}$ ... $(Y_i$ μ_i $(Y_j$ μ_j D, X, A $1_i(t$ $1_j(t$ $($

$\hat{p}_t(i, X, A$

$\Delta_i(t \quad (\hat{\mu}_t(i \quad \mu_t(i \quad p_t(i \quad 1_i(t$

GNNs for Network Confounding

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