1 Introduction

Ten thousand children need to be allocated into ten schools, each accommodating one thousand of them. The schools are not the same, and parents may rank them in different ways. However, if all children are considered equal, then a social lottery, where each student has an equal chance to attend each of the ten schools, seems to be the best solution.[1] This procedure is egalitarian — everyone gets the same lottery — and feasible. But is it efficient? Specifically, is there no other procedure such that ex-ante, before people know their allocated school, they will get a better lottery?

If individual preferences over the schools are not the same, then this procedure may be inefficient, for example, if each school is ranked best by exactly 1000 parents. It is true that if all individuals are expected utility maximizers and have the same preferences over lotteries (and in particular, over the schools), then this procedure leads to an efficient allocation. This is also the case if all have the same quasi-concave preferences, i.e. preferences for randomization over lotteries. But if preferences are quasi-convex, and a mixture of two indifferent lotteries is inferior to the mixed lotteries, then we show that this procedure is never efficient, regardless of whether individual preferences are the same or not. Such preferences are implied by some well known alternatives to expected utility theory (for example, Tversky and Kahneman's [39] Cumulative Prospect Theory, where risk aversion implies quasi-convexity. See discussion below).

We analyze first an economy where N types of goods with k units each need to be allocated, one for each of Nk agents. All agents have strict preferences over the basic goods, and continuous, monotone (with respect to first-order stochastic dominance), and strictly quasi-convex preferences over

[1]For example, divide the students into ten groups $A_1, \ldots, A_{10}$ of size 1000 each. Choose with probability $\frac{1}{10}$ one of the ten permutations $\sigma_1, \ldots, \sigma_{10}$ of $(1, \ldots, 10)$, where $\sigma_j(i) = (i + j - 1) \pmod{10} + 1$, $j = 1, \ldots, 10$.

such lotteries by multiplying the probabilities of the two stages, this extra randomization will reduce participants utilities. But if decision makers instead satisfy the compound independence axiom, according to which if they prefer q to q^0 they will prefer to replace q^0 with q in any compound lottery that includes the former as an outcome, then such randomizations will not change agents' welfare.

Our analysis depends on the assumption that individual preferences over lotteries are quasi-convex. Expected utility, where preferences are linear in the probabilities, is the boundary case. Strict quasi-convexity is for example the case with the popular family of rank-dependent utilities models (Quiggin [31]), which also includes Yaari's [46] dual theory, as well as Tversky and Kahneman's [39] Cumulative Prospect Theory, where risk aversion implies quasi-convexity. Other models which can exhibit quasi-convexity include quadratic utility (Chew, Epstein, and Segal [11]), and Kőszegi and Rabin's [24] models of reference-dependence. In addition, Machina [25] pointed out that quasi-convexity occurs if, as is common in many applications such as insurance purchasing, before the lottery is resolved agents can take actions that a ect their nal utility. If the optimal action depends on the probabilities, the induced maximum expected utility will be convex in the probabilities, meaning that even if the underlying preferences are expected utility, induced preferences over the optimal lotteries will be quasi-convex.

The experimental evidence on quasi-convexity versus quasi-concavity is mixed. Most of the experimental literature that documents violations of expected utility (e.g., Coombs and Huang [13]) found either preference for randomization or aversion to it. Camerer and Ho [9] nd support for quasi-convexity over gains and quasi-concavity over losses. An example of behavior that distinguishes between the two attitudes to mixture is the probabilistic insurance problem of Kahneman and Tversky [23]. They showed that in contrast with experimental evidence, any risk averse expected utility maximizer must prefer probabilistic insurance to regular insurance. Sarver [33] pointed out that this result readily extends to the case of quasi-concave preferences. In contrast, quasi-convex preferences can accommodate aversion to probabilistic insurance together with risk aversion (for example, risk-averse rank-dependent utility; see Segal [34]). Sarver further illustrates that quasi-convex preferences are consistent with increasing marginal willingness to pay for insurance at some levels of coverage; another plausible property that in most models requires violation of risk aversion. In the context of group decision making, Dillenberger and Raymond [17] show that quasi-convexity of preferences in the individual level is equivalent to the consensus effect: individuals tend to conform to the choices of others in group decisions, compared to choices made in isolation.

The idea of using lotteries to allocate indivisible goods is not new (see, for example, Diamond [14], Hylland and Zeckhauser [22], and Rogerson [32]). Moreover, the *possible* existence of an optimal solution that induces each individual to face a binary lottery was already discussed in Hylland and Zeckhauser [22] under expected utility preferences. Our approach differs from these works. We show that in a large economy with quasi-convex preferences, any ex-ante efficient solution *must* use only binary lotteries. Also, as long as individuals simplify compound lotteries by multiplying the probabilities, randomizing among these binary lotteries (thus giving identical people the same ex-ante lottery) is always suboptimal.

In this paper we employ a strong notion of ex-ante efficiency, which takes into consideration individual preferences over lotteries. Two weaker notions of efficiency were previously studied, ordinal efficiency and ex-post efficiency, both only depend on ordinal rankings of the final goods. As we remark in Section 2.1, our results imply that many of the popular allocation mechanisms used in the literature are ex-ante inefficient. For example, random serial dictatorship, that assigns the order of individuals using uniform distribution, is inefficient as it typically implies that each individual will face a lottery with more than two elements in its support. Note that this inefficiency relies only on the ordinal property of the preferences over lotteries, namely that they are quasi-convex in probabilities.

The stronger notion of efficiency we consider, which is natural once individuals preferences over lotteries are taken into account, makes it harder to achieve strategy proofness, a property that ensures that it is always optimal for agents to truthfully report their preferences over lotteries. This raises the questions of who can use our results and how. We discuss this issue at length in Section 4. While not always possible, we argue that in many situations social planners can collect at least partial information about cardinal properties of preferences, and our results can guide them how to locally improve upon existing popular methods (a similar approach was suggested by Abdulkadiroglu, Che, and Yasuda [2]). Moreover, empirical and experimental data regarding individual preferences can be collected and used in order to estimate ideal solutions. Such methods are used in various situations, for example, in medical decision making (Wakker [44]).

The paper is organized as follows. Section 2 lays out the basic problem in a finite environment and states our main results. Section 3 studies two possible extensions: the case where the number of agents and units is not the same, and the case of a continuum economy. Section 4 comments on the applicability of our approach. In Section 5 we discuss the benefit of a pre-randomization over the allocation lotteries. Section 6 concludes with a further discussion of binary lotteries and the applicability of our results. All proofs are in the Appendix.

2 Finite Economies

Consider an economy with Nk individuals and with k units of each of $N > 3$

$>$

of notation, we identify x_i with the lottery that yields it with probability 1. Each member n of society has preferences $_n$ over such lotteries,

2.1 Ex-Ante Efficiency

We first characterize solutions **q** that are feasible, that is, satisfy equations (1) and (2), and are ex-ante Pareto efficient, in the sense that there is no solution $\tilde{\mathbf{q}}$ such that $\tilde{q}^n \succsim q^n$ for all n and $\tilde{q}^m \succ q^m$ for some m. As preferences are continuous over a compact domain, feasible efficient allocations exist. We show that in such allocations, and without any further assumptions on individual preferences, all but `not too many' individuals obtain either a degenerate lottery or a lottery with positive probabilities on two goods only.

Definition 1 A lottery q^n is binary if $q_i^n > 0$ for *no more than* two outcomes.

Theorem 1 Let **q** be a feasible and Pareto efficient solution. Then for any three goods x_r, x_s, x_t, there is at most one person n such that $q_r^n, q_s^n, q_t^n > 0$.

This result implies that to detect violations of ex-ante efficiency, it is enough to observe an allocation in which two individuals receive lotteries that put positive probabilities on the same three goods. The exact probabilities are inconsequential.

To illustrate the main argument of the theorem, suppose that two agents m and n

generality, one of the supporting slopes to the indi erence curve of person n through $(q_t^n; q_r^n)$ is weakly steeper than one of the supporting slopes to the indi erence curve of person m through $(q_t^m; q_r^m)$. Take a line with slope between these two values. To make both agents better o , transfer probabilities from one agent to another as depicted in the gure, a violation of the e ciency assumption. If, instead, individuals' ordinal rankings of the goods are *not* identical, then the two agents can trade in the probabilities of any two goods that they rank di erently to improve ex-ante welfare.

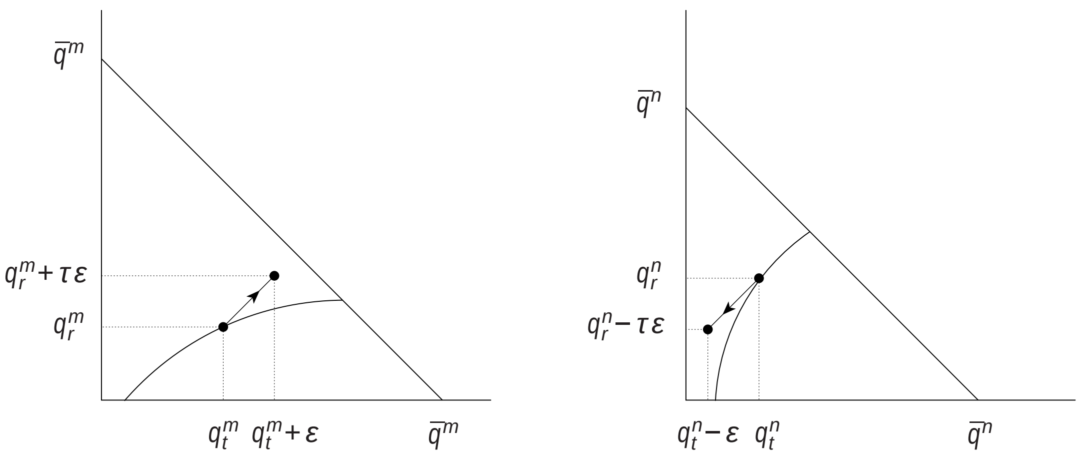

Figure 1: Changes in the allocations of individuals m and n

As we explain below, the arguments above also apply to agents with di erent expected utility preferences.

Theorem 1 implies a limit on the number of individuals who can receive a non-binary lottery.

Corollary 1 The number of individuals who hold non-binary lotteries in any feasible and e cient allocation is bounded above by $\frac{N}{3}$.

The number of subsets of $\{1, \ldots, N\}$ where no two elements have an intersection with more than two numbers is bounded above by $\binom{N}{3}$, which is the case where all subsets have three elements each.[4] Since the number of individuals who hold non-binary lotteries is bounded above by $\binom{N}{3}$ while the total population size is Nk, their fraction becomes arbitrarily small as k increases.

While for exposition purposes we confine our attention to the case of strict quasi-convex preferences, Theorem 1 generally also holds under expected utility, which is linear (and hence also weakly quasi-convex) in probabilities.[5] Under expected utility, if all agents have the same preferences over lotteries, then there are many efficient solutions, including interior ones. Our results are thus more prominent once preferences are cardinally different. More precisely,

Proposition 1 Consider two expected utility agents m and n with utility functions over final outcomes u_m and u_n, respectively. For any three goods x_r, x_s, and x_t, if $\mathbf{q}$ is a feasible allocation with both $q_r^n; q_s^n; q_t^n > 0$ and $q_r^m; q_s^m; q_t^m > 0$, and if

$$\frac{u_m(x_s) - u_m(x_t)}{u_m(x_r) - u_m(x_s)} \neq \frac{u_n(x_s) - u_n(x_t)}{u_n(x_r) - u_n(x_s)}$$

then $\mathbf{q}$ is inefficient ex-ante.

In words, if the slopes of the two agents' indifference curves in the corresponding probability triangles are not the same, then any allocation that gives both agents lotteries with positive probabilities on these three goods

[4] This bound may be tighter under further assumptions on individual preferences. See for example the case of same preferences in Section 2.2.

[5] Assuming that all individuals are expected utility maximizers, Hylland and Zeckhauser [22] use competitive equilibrium with equal incomes to show the existence of a solution in which almost all agents receive a binary lottery. Our result holds without relying on any market mechanism.

is ine cient. The proof is identical to the one given in the appendix for Theorem 1 and is omitted.

There are many popular mechanisms that can be used to allocate objects among a group of agents. One example that is broadly used and is easy to implement is random serial dictatorship. Randomly order the Nk individu-

Importantly, this argument only relies on simple, observable information: quasi-convexity of preferences and the size of the supports of the lotteries that are used.

Assuming expected utility, Bogomolnaia and Moulin [7] show how random serial dictatorship, which uses uniform distribution to rank agents, is not necessarily even ordinally e cient, as it may induce for each agent a distribution over the goods that is stochastically dominated, with respect to that agent's ordinal preferences, by another feasible distribution. Their sug-

Theorem 2 Suppose that $_1 = \ldots = _{Nk} =$. Then:

1. Ideal solutions exist.
2. The number of different binary lotteries used in any ideal solution is bounded above by $M = \left\lfloor \frac{N^2}{4} \right\rfloor$.
3. An ideal solution yields all but at most $M = \left\lfloor \frac{N^2(N-2)}{8} \right\rfloor$ agents a binary lottery.

Below, we outline the main steps involved in the proof.

For part 1, let

2.

used involves outcomes x_i and x_j with $i < j$, then since all lotteries on outcomes better than x_i dominate it and all lotteries on outcomes inferior to x_j are dominated by it, such lotteries cannot be part of the ideal solution.

Similarly, the bound on the number of agents who hold non-binary lotteries (which for $N > 4$ is lower than the $\frac{N}{3}$ bound from the general case of Theorem 1) is the number of non-dominated lotteries with three possible outcomes that can simultaneously be used. Note that many individuals may hold the same binary lottery, but only one individual can hold any non-binary lottery.

The proofs of parts 2 and 3 of Theorem 2 only use the requirement that the lottery received by one person cannot dominate the lottery received by another. The actual number of binary lotteries used in an ideal solution can be much smaller than the upper bound suggested by the theorem. Theorem 3 of Section 3.1 identifies conditions under which the set of binary lotteries in **q** is either $\{(q_1; q_i)\}_{i=2}^{N}$ or $\{(q_i; q_N)\}_{i=1}^{N-1}$. The number of binary lotteries used in these cases is $N - 1$, significantly less than the bound obtained in part 2 of Theorem 2. For example, for $N = 10$, the conditions of Theorem 3 imply 9 binary lotteries, whereas the bound of Theorem 2 is 25. Note that the lower bound on the number of binary lotteries to be used is $\lceil \frac{N}{2} \rceil$, where $\lceil x \rceil$, the ceiling of the real number x, is the lowest integer greater than or equal to x. This will be the case when a feasible solution is obtained by a set of lotteries $(q_1; q_N); (q_2; q_{N-1}); \ldots$ that are all equally attractive in .

3 Extensions

We discuss two possible extensions to our basic framework. We first analyze assignment problems where all agents have the same preferences, but the number of units to be allocated is not equal to the number of agents. Second, we consider a continuum economy with the same mass of agents and goods.

3.1 Di erent Number of Units and Agents

Consider again the case where all agents have the same preferences over lotteries (as in section 2.2) and suppose as before that x_1

Theorem 3 Suppose that all Nk agents have the same preferences and that $x_1 \succ \cdots \succ x_N$. If x_N is a terrible outcome, then all the binary lotteries used by an ideal allocation **q** are of the form $(q_i; q_N)$, $i = 1, \ldots, N - 1$, and for a sufficiently large k, they are all used. Parallel results hold for the case where x_1 is an excellent outcome, with the binary lotteries $(q_1; q_i)$, $i = 2, \ldots, N$.

If there are more agents than units, define a new good x_{N+1} which is "receive nothing." At least in the case of school allocation, this may well be a terrible outcome. Theorem 3 implies that in that case, almost all children face a lottery where there are two possible outcomes: either they go to a specific school, or they stay at home. In other words, they face uncertainty regarding acceptance, but not regarding the school into which they will be accepted. Equality implies that the better the school, the less likely is a holder of a lottery for this school going to win.

3.2 Continuum Economies

Consider a continuum economy with a unit mass A of N equally sized (with respect to the Lebesgue measure λ) types of agents $A_1, \ldots, A_N$. There is a unit mass B of N goods $x_1, \ldots, x_N$ to be allocated among them, where the mass of each unit is $\frac{1}{N}$.[8] Each of the individuals of type i has strictly quasi-convex preferences $\succsim_i$ over lotteries over the N goods.

Our aim in this paper is to analyze possible mechanisms for the allocation of goods which are desired by all, as otherwise there is no need for a

[8] In fact, we can assume J types of goods, and that both the N types of individuals, as well as the J types of goods, are not of same size. However if the sizes are rational numbers, we can assume without loss of generality that $J = N$ and the sizes of the different goods are the same; and if they are irrational, we'll obtain our results using continuity, where the economy is the limit of economies with rational sizes. We therefore assume throughout $J = N$ and that the sizes of the types of agents and of the goods are all $\frac{1}{N}$. See Footnote 10 below for a further generalization.

Maniquet [26]). We show however that in the present context, the continuum economy guarantees the existence of no-envy allocations.

Theorem 4 A feasible and efficient allocation for the continuum economy with strictly quasi-convex preferences yields all agents a binary lottery. The set of no-envy such allocations is not empty, and if all agents have the same preferences, then equality with such lotteries is obtained.

We offer here an outline of the proof. The first step shows, similarly to the proof of Theorem 1, that efficient allocations must yield all agents a binary lottery. Next, we start from an allocation where everyone is facing the lottery that gives them an equal chance for each of the goods and employ a known technique of demand-sets convexification (see Mas-Colell, Whinston, and Green [28, Section 17.I] which is based on Starr [37]) to obtain a competitive market equilibrium prices and allocations. Given these prices, all agents will maximize their utility along the same budget set, so No-Envy is guaranteed. Competitive equilibria are feasible and efficient, hence the claim of the theorem.

There is however one issue that requires special attention in which our analysis of the market equilibrium differs from the literature. Formally, the lottery $(x_1, q_1; \ldots; x_N, 1 - \sum_{i=1}^{N-1} q_i)$ is represented as the vector $(q_1, \ldots, q_{N-1})$ in the $N-1$-dimensional simplex. This is different from the standard model, where the domain of preferences is not bounded from above. To see why this may create a problem, consider Example 1 in the Appendix with $N = 3$ where $x_1 \succ x_2 \succ x_3$. The preferences of this example are monotonic in the probabilities q_1 and q_2 in the sense that if $(q_1', q_2') \gneqq (q_1, q_2)$, then $(q_1', q_2') \succ (q_1, q_2)$. But they do not satisfy monotonicity with respect to first order stochastic dominance, in the sense that for $\varepsilon > 0$, $(q_1 + \varepsilon, q_2 - \varepsilon) \prec (q_1, q_2)$, and equilibrium does not exist. We show in the proof of Theorem 4 that this stronger version of monotonicity eliminates the existence problem.

Remark 1 Let T be the number of lotteries used in the proposed solution. Then for $h = 1, 2, \ldots, T$ there is a continuum of agents who receive the same binary lottery, say $(x^h, \;{}^h; y^h, 1 \quad {}^h)$ for some outcomes x^h, y^h and ${}^h \; 2 \; [0, 1]$. The implementation of this, so that the fraction of the people in this group that receives x^h is ^{h}, can be guaranteed by using the appropriate law of large numbers for a continuum of independent random variables. Such approach appears, for example, in Sun [38], and we adopt here his measure theoretic framework.[10]

4 Is the Data Available?

of Camerer [8] and Starmer [36]). One of the most popular group of models is based on the idea that the evaluation of the probability of an outcome depends on its rank in the support of the lottery. This family includes Quiggin's [31] rank-dependent utilities, Yaari's [46] dual theory, and Tversky and Kahneman's [39] cumulative prospect theory. For $x_1 \geq x_2 \geq \dots \geq x_N$, the rank dependent functional form is given by

$$V(q) = u(x_1)\varphi(q_1) + \sum_{i=2}^{N} u(x_i)\left[\varphi\left(\sum_{j=1}^{i} q_j\right) - \varphi\left(\sum_{j=1}^{i-1} p_j\right)\right]$$

where $\varphi : [0,1] \to [0,1]$ is strictly increasing and onto. To use the rank dependent model, one needs to know the utilities from the outcomes and the transformation function the decision maker is using to evaluate the (cumulative) probabilities.

practically used to improve medical decisions (Wakker [44]). It is well known that measures of risk aversion tend to be context-speci c and vary across domains (see, among others, Weber, Blais, and Betz [45] or Hanoch, Johnson, and Wilke [20]). Yet, and even though there is not yet a consensus about the \best" approach to use (Charness, Gneezy, and Imas [10]), methods analogous to those employed in nancial, health/safety, recreational, ethical, and social decisions (as in Weber, Blais, and Betz [45], or the ones discussed in Finkelstein, Luttmer, and Notowidigdo [18] to estimate health state dependence of the utility functions) can be used in the domain of lotteries over apartments or schools.

Even without any information about the speci c model used by members of society our results suggest ways for welfare improvements. Sometimes the social planner has information about other characteristics of the agents that can be used to assess their intensity of preferences over allocations. For example, it is plausible that a resident of a certain neighborhood would put higher premium on attending a school in close proximity compared to someone who considers only remote schools. Similarly, a religious person will naturally have stronger preferences for schools that have religious components in their operations or curriculum compared to someone who does not take this dimension into consideration.

Such information can be used in the following way. Starting from an allocation that results from a strictly strategy-proof mechanism with respect to the ordinal rankings (for example, random serial dictatorship), ex-ante allocations may yield two agents positive probabilities overwfrom a (it enc](iv)2l8.058r77()-3iomation preferences and large economies, the authors offered a mechanism that allows students to signal their cardinal preferences, and showed that it (ex-ante) Pareto dominates the popular deferred acceptance mechanism. Their new mechanism is typically ex-ante inefficient and is only ordinally strategy-proof.

5 Ex-ante Lotteries

If preferences are strictly quasi-convex, then giving two identical agents the same interior outcome must be inefficient, as moving in opposite directions along a supporting plane of the indifference curve will make both better off. Instead of equality in outcomes, allocation mechanisms will seek a weaker notion of equality, where identical agents will be indifferent between their respective outcomes. This is indeed the conclusion from Theorem 2, where everyone is indifferent between all allocated lotteries, even though they are not the same.

of Theorem 2 are allocated by a lottery.[12] Given that all individuals will face the exact same lottery, this procedure guarantees full equality in the ex-ante stage.

The e ectiveness of this procedure crucially depends on the agents' attitude towards multi-stage lotteries. Denote the relevant binary lotteries $q^{(1)}$

it is yet to be examined in scrutiny for the specific context of allocation mechanisms.[13]

6 Concluding Remarks

The use of binary lotteries is pervasive in economics. Many experimental works are conducted with choices among such lotteries (or between them and sure outcomes), where the main rationale for using binary lotteries is that they are easily interpretable. Some recent theoretical papers use simplicity criteria to argue for the attractiveness of binary lotteries in terms of minimizing complexity costs (for example, Puri [30]), and of binary acts, that are always `well-understood' and can be used as a tool for making difficult comparisons (Valenzuela-Stookey [40]).

In our setting, that (almost) everyone should receive a binary lottery follows mathematically from the assumption that all individual preferences are quasi-convex. As argued above, this gives us a simple necessary condition that can be used to assess whether an allocation of lotteries is ex-ante efficient. But as a social mechanism, binary lotteries have another independent attraction of their own. When facing a lottery over a set of outcomes on which they do not have full information, it is quite natural for people to look for such information before the lottery is played. If so, it is clearly better for them to face a lottery with fewer outcomes.

stage, when they'll face a lottery over two outcomes only.

Our aim in this paper is to suggest a new way to assess and think of they can be done without violating eqs. (1) and (2). In both panels, the probability of x_t is measured on the horizontal axis and that of x_r on the vertical one. The only values of $\mathbf{q}$ that will be changed are those of q_i^a for $a = m; n$ and $i = r; s; t$. We will therefore deal with the induced preferences over the above triangles and ignore the rest of the probabilities. To simplify notation, we write $(q_t^a; q_r^a)$ for $(q_t^a; q^a \quad q_t^a \quad q^a$

vector in the compact probabilities simplex Δ^{N-1}, hence it follows by standard arguments that there is a subsequence of $\mathbf{q}^h$, without loss of generality the sequence itself, such that for all $n = 1,\dots,Nk$, $q^{n,h} \to q^{n,*}$. The vector $\mathbf{q}^* = (q^{1,*},\dots,q^{Nk,*})$ satisfies eqs. (1) and (2), hence it is a solution. Since V is continuous it satisfies equality, and by the continuity of V, $V(q^{n,*}) = v$. It follows by the definition of v that for any solution $\mathbf{q} = (q^1,\dots,q^{Nk})$ satisfying equality, $q^{n,*} \succeq q^n$, $n = 1,\dots,Nk$.

Lemma 2 Let $\mathbf{q}$ be a feasible solution in which for some two individuals m and n, $q^n \succ q^m$. Then there is a feasible solution $\mathbf{q}'$ where $q^n \succ q'^n \succ q'^m \succ q^m$, and for $\ell \neq n,m$, $q'^\ell = q^\ell$.

Proof: Since $q^n \succ q^m$, it follows by monotonicity with respect to first-order stochastic dominance (in short, by FOSD) that there are goods r and s such that $x_r \succ x_s$ and such that $\varepsilon = \min\{q^n_r, q^m_s\} > 0$, as otherwise $q^m \succeq q^n$. In both profiles below, q^ℓ does not change for all $\ell \neq n,m$. For $\varepsilon' \leqslant \varepsilon$, let $q'^n = (q^n_r - \varepsilon', q^n_s + \varepsilon', q^n_{-r,s})$ and $q'^m = (q^m_r + \varepsilon', q^m_s - \varepsilon', q^m_{-r,s})$. For a sufficiently small $\varepsilon' > 0$, $q^n \succ q'^n \succ q'^m \succ q^m$.

Lemma 3 The solution $\mathbf{q}^*$ as in Lemma 1 is efficient.

Proof: Let $\mathbf{q}^*$ be as in Lemma 1, and suppose that there is $\bar{\mathbf{q}} = (\bar{q}^1,\dots,\bar{q}^{Nk})$ such that wlog $V(\bar{q}^1) \geqslant \dots \geqslant V(\bar{q}^{Nk}) \geqslant V(q^{1,*}) = \dots = V(q^{Nk,*})$, where at least one of these inequalities is strict. Applying Lemma 2 $Nk - 1$ times at most, we can create a feasible allocation $\mathbf{q}$ such that for all n, $V(q^n) > V(q^{1,*})$. Let $w = \min\{V(q^n)\}$ and define

$$b = \inf\left\{\max_n\{V(q^n)\} - \min_n\{V(q^n)\} : \mathbf{q} \text{ is feasible and } \min_n\{V(q^n)\} > w\right\}$$

As in the proof of Lemma 1, there is a feasible solution $\hat{\mathbf{q}}$ for which b is obtained. By Lemma 2, $b = 0$. This means that $\hat{\mathbf{q}}$ satisfies equality, a contradiction to the definition of $\mathbf{q}^*$.

By Lemma 1, the feasible solution $\mathbf{q}^*$ satisfies equality, and by Lemma 3 it is efficient, hence it is an ideal solution.

2. The number of different binary lotteries used in any ideal solution is bounded above by $M = \left\lfloor \frac{N^2}{4} \right\rfloor$: Let B be the set of binary lotteries used by an ideal solution $\mathbf{q}$. We show that there is t^* such that for all non-degenerate $(q_r; q_s) \in B$, $r \leq t^* < s$.

Case 1: If one of the lotteries in B is degenerate, say δ_{t^*}, then by equality and FOSD, for any $(q_r; q_s) \in B$ it must be the case that $r < t^* < s$.

Case 2: There is no t for which there exists $(q_r; q_s) \in B$ such that $t < r < s$

non-binary lotteries allocated by it. That is, $C = \{\{r_1 < r_2 < \ldots < r_d\} : (q_{r_1}, q_{r_2}, \ldots, q_{r_d})$ is one of the lotteries allocated by $\mathbf{q}\}$.

Similarly to part 2 above, we show that there is ℓ such that for all $\{r_1 < \ldots < r_d\} \in C$, $r_1 \leq \ell < r_d$. If there is no ℓ for which there exists $\{r_1 < \ldots < r_d\} \in C$ such that $\ell < r_1$, then all lotteries with indexes in C must have x_1 as one of their outcomes, in which case we set

Proof of Theorem 3: Denote by q^T and q^E the degenerate lotteries that yield x_N and x_1 with probability 1, respectively. For q, q', let $[q, q'] = \{\ q + (1\ \ )q' : \ \in [0, 1]\}$. A set A of lotteries is above set B if for all $q \in A$, $[q, q^T] \cap B \neq \varnothing$. It is below B if for all $q \in A$, $[q, q^E] \cap B \neq \varnothing$. Since H^T is the convex hull of points $\{(q_i, q_N)\}_{i=1}^{N-1}$, every lottery q is either above or below H^T. By quasi-convexity, if $q \quad x_{N-1}$ then q is above H^T, hence so is q such that $q \quad x_{N-1}$. By FOSD, for $i > j \neq N$ with $q_i > 0$, $(q_i, q_j) \quad x_{N-1}$. If it is part of a solution **q**

orem 1, where individuals m and n are replaced with A_1 and A_2. It follows that all agents receive a binary lottery.

Let $\ ^{N\ \ 1} = f(\ _{1}; : : : ;\ _{N\ \ 1})\ 2\ <^{N\ \ 1}_{+} :\ \mathrm{P}^{N\ \ 1}_{i=1}\ \ _{i} = 1g$ be a prices simplex. For every $\ 2\ \ ^{N\ \ 1}$, $\ (^{1}$

correspondences of the various agents are given by

$$D_1(\ ) = D_2(\ ) = \begin{cases} (1;0) & 6\ \frac{1}{2} \\ (\frac{1+}{3};0) & \frac{1}{2} < \quad < 3 \\ f(\frac{3+5}{18}; {}^{5(1} & \end{cases}$$

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