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Transport in Transitory Dynamical Systems

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1. Transitory systems.

Invariant manifolds have long been recognized as important structures that govern global behavior in dynamical systems. Hyperbolic manifolds in particular, by their very definition, relate information about the exponential contraction and expansion of nearby trajectories within the flow and so play crucial roles in the dynamics of such systems, lending insight into the mechanisms by which chaos, mixing, transport, and other complex global phenomena occur. Transverse intersections of stable and unstable manifolds give rise to lobes defining of packets of trajectories that exit or enter coherent structures or resonance zones bounded by pieces of the manifolds. Thus, tracking these lobes provides a means for quantifying flux between coherent structures in the flow.

The treatment of mixing and transport for aperiodically time-dependent flows, however, requires the development of new methods because the concept of invariance may be too strong and may not even lead to physically relevant structures. One popular method, the finite-time Lyapunov exponent (FTLE), has been used extensively in recent years to compute local approximations of invariant manifolds and identify structures that remain coherent in the Lagrangian sense on some finite time interval [21, 52, 30]. Another idea uses a nonautonomous

While these Lagrangian frameworks for identifying coherent structures in nonstationary systems have seen broad application, using them to accurately quantify transport and mixing over finite timescales remains a di cult task. A few recent studies have managed to give numerical estimates for finite-time transport and mixing within aperiodic time-dependent flows [47, 11, 8, 43]. In addition, the instantaneous flux across a "gate surface" can also be estimated using only local Eulerian information [20, 4], and the results have been applied to geophysical flows defined both analytically and by discrete data sets. One such application is the investigation of eddy-jet interactions, in which the strength of the interaction is quantified by the amount of fluid entrained by or ejected from the eddy [46, 19]. However, the accuracy of these methods is tied to the rate of change of the Eulerian velocity field, and, moreover, the instantaneous flux through a surface does not provide an estimate for the finite-time transport between two disjoint coherent structures.

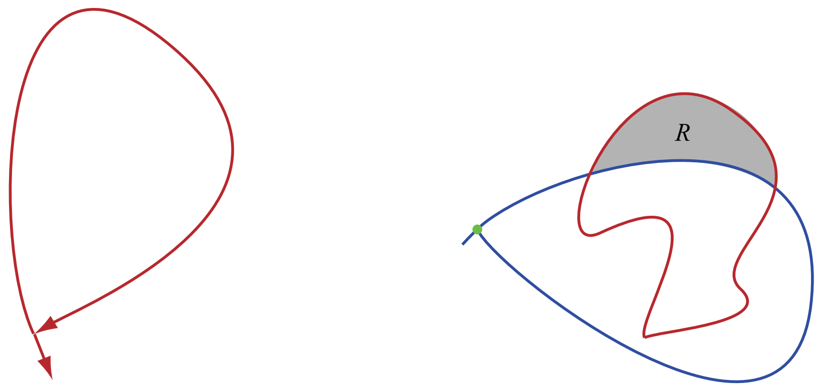
Since the mere identification of Lagrangian coherent structures (LCSs) in fully aperiodically time-dependent systems is a challenging problem itself, we study here a simpler problem—the quantification of finite-time transport between coherent structures in two-dimensional systems undergoing a transition between two steady states. Our methods are Lagrangian, in the sense that they rely on knowing certain key trajectories, and as such they

In this paper we are interested in quantifying the transport between coherent structures of the past and future vector fields P and F for two-dimensional systems of the form (1.1). To define these, it is natural to consider hyperbolic orbits of P and F and their stable and unstable manifolds since initial segments of these often define invariant or nearly invariant structures such as *resonance zones* [39, 13, 34]. However, determining which structures of P and F are relevant to the dynamics of the full nonautonomous vector field V requires special attention.

It is natural to think of the dynamics of (1.1) as occurring on the extended phase space $M \times \mathbb{R}$. We will assume that V has a complete flow, $\phi_{t_1,t_0} : M \to M$ for any $t_0, t_1 \ \mathbb{R}$, where ϕ_{t_1,t_0} maps a point from its position at $t = t_0$ to its position at $t = t_1$. Then every point $(x, t) \ M \times \mathbb{R}$ has an orbit

$$\gamma(x, t_0) \equiv \{(\phi_{t_1,t_0}(x), t_1) : t_1 \ \mathbb{R}\} \ M \times \mathbb{R},$$

and such sets are invariant in the sense that for any $\tau \ \mathbb{R}$, $\gamma(x, t) = \gamma(\phi_{\tau,t}(x), \tau)$. Of course, this notion of invariance is not restrictive: any subset $S \ M$ can be taken to be the time-τ slice through an invariant set $\gamma(S, \tau) = \{(x, t) : x \ \phi$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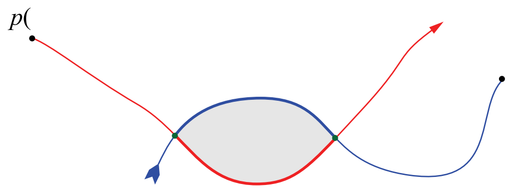
will think of LCSs as being objects bounded by separatrices of P for $t < 0$ or of F for $t > \tau$. These structures may also be ephemeral under the vector field V; however, for a transitory system we think of only one event, encapsulated by the transition map T, as creating or destroying an LCS.

In Figure 1, the coherent structure bounded by Γ_p in the past vector field P is destroyed by the transition map T, giving rise to a new structure bounded by Γ_f in the future vector field F. There are two heteroclinic points $\{h_1, h_2\} = T(\Gamma_p) \cap \Gamma_f$ in the time-τ slice; they are backward-asymptotic to p, forward-asymptotic to (Γ


p(

Consequently

$$\begin{aligned}\int_U \nu &= \int_t^\tau \left(\int_U d(\phi_{s,\tau} L)\right) ds + \int_U \phi_{t,\tau}\nu \\ &= \int_t^\tau \left(\int_{\phi_{s,\tau}(U)} dL\right) ds + \int_{\phi_{t,\tau}(U)} \nu \\ &= \int_t^\tau \left[L(h_1(s), s) - L(h_0(s), s)\right] ds + \int_{\phi_{t,\tau}(U)} \nu. \end{aligned} \tag{2.5}$$

Since $h_0(t) \to h_1(t)$, the length $|\phi_{t,\tau}(U)| \to 0$ as $t \to -\infty$. Taking this limit yields the result (2.4). ■

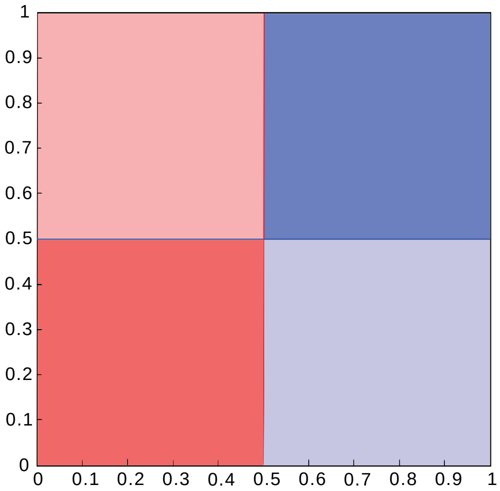
We note that $\Delta A_\tau^-(h_0, h_1)$ in (2.4

Similarly, the stable manifold of $(1,0)$ is a subset of its unstable manifold; the slices at $t=\tau$ are

$$W_\tau(1,0) = \{(1,y) : 0 < y < \tfrac{1}{2}\} \qquad W_\tau(1,0) = \{(1,y) : 0 < y < 1\}.$$

Thus, though $(0,1)$ and $(1,0)$ are both forward and backward hyperbolic, in the sense of Definition 1.2, neither is a hyperbolic orbit of V.

The past vector field also has two saddle equilibria at $p_0 = (\frac{1}{2}, 0)$ and $p_1 = (\frac{1}{2}, 1)$, and the future vector field has saddles at $f_0 = (0, \frac{1}{2})$ and $f_1 = (1, \frac{1}{2})$. Under (3.1) the orbits of these points remain on the invariant boundaries of M, and, as we shall see, these orbits play crucial roles in the delineation of Lagrangian [illegible]


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

reversed by R , the push-forward (B.3) of R:

$$DR\,V(y, x, \tau - t) = -V(x, y, t).$$

Consequently, R inverts the transition map, $T^{-1} = R \circ T \circ R$, and since $R($

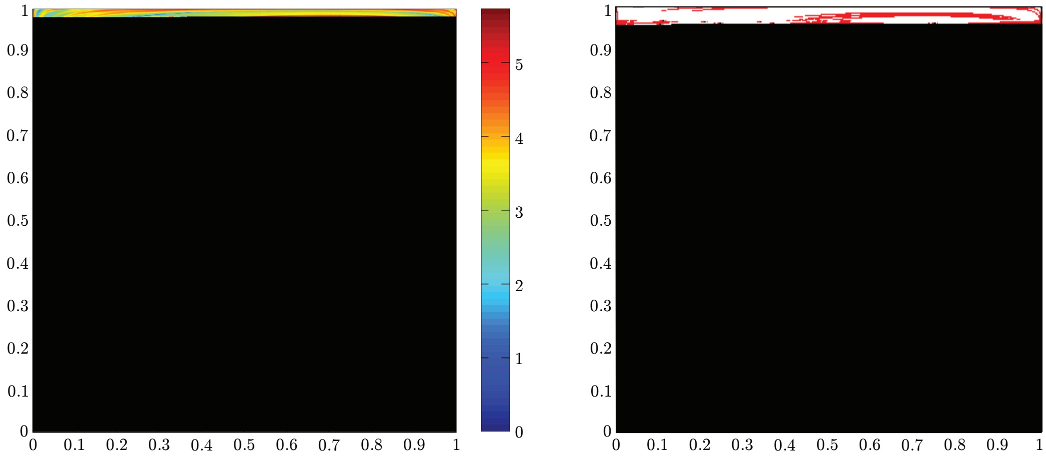
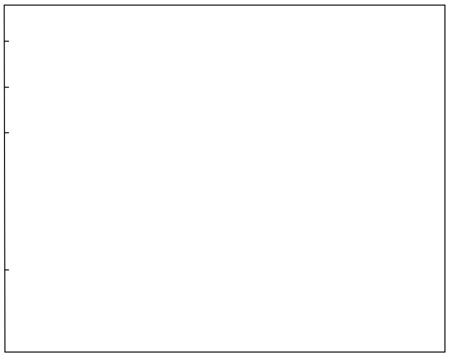
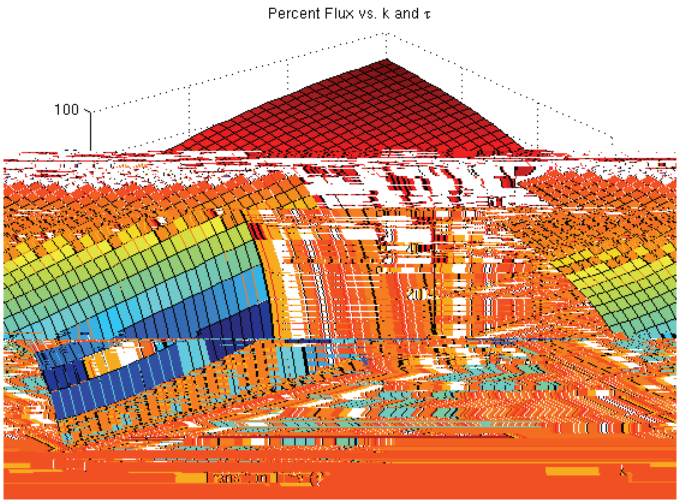


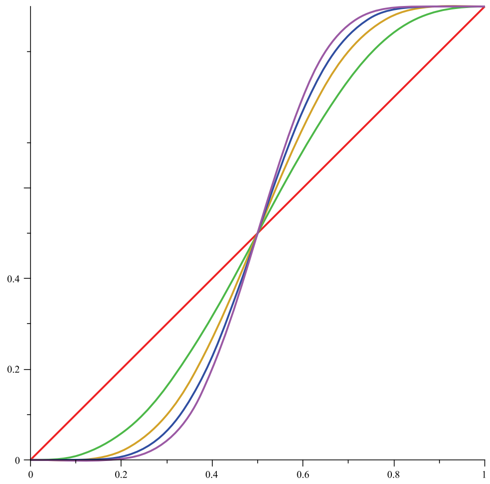
Figure 8. *Backward time FTLE field for* (3.1) *at the transition time* $\tau = 0.8$ *using a backward integration time of* 1.2 *and a* 1500×1500




Percent Flux vs. k and τ
100

4. Conclusions. While techniques involving finite-time Lyapunov exponents and distinguished hyperbolic trajectories have recently been developed for the identification and extraction of coherent structures in time-dependent systems, they have been used only selectively to give quantitative descriptions of the finite-time flux between such structures. One reason for this is that the "ridges" of the FTLE field that represent approximate invariant manifolds are often di cult to extract, making precise measurements of flux challenging.

Here we have considered a special class of two-dimensional nonautonomous systems that exhibit time-dependent behavior only on a compact interval and have extensively used the concepts of backward- and forward-hyperbolicity for these transitory systems. The special structure of these systems leads to a simple method for the numerical computation of flux between LCSs in the Hamiltonian case. Our method relies primarily on knowledge of heteroclinic orbits and their associated invariant manifolds that bound lobes within the extended phase space. Thus, our computations of flux require very little Lagrangian information relative to computations involving FTLE or distinguished hyperbolic trajectories. In particular, our adaptive computation of $T(U)$ allowed for an order of magnitude reduction in the number


0
0.2
0.4
0
0.2
0.4
0.6
0.8
1

where $\cdot$ is any tensor. In particular, for a vector field X,

$$L_V X = [V, X] = (V \cdot \nabla)X - (X \cdot \nabla)V, \tag{B.6}$$

where [,] is the Lie bracket. The Lie derivative acting on differential forms obeys Cartan's homotopy formula

$$L_V \alpha \equiv \imath_V(d\alpha) + d(\imath_V \alpha). \tag{B.7}$$

Note that L behaves "naturally" with respect to the pullback:

$$f^* L_V \alpha = L_{f^* V} f^* \alpha. \tag{B.8}$$

Acknowledgments.