Stability of minimal periodic orbits

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Abstract

Symplectic twist maps are obtained from a Lagrangian variational principle. It is well known that nondegenerate minima of the action correspond to hyperbolic orbits of the map when the twist is negative definite and the map is two-dimensional. We show that for more than two dimensions, periodic orbits with minimal action in symplectic twist maps with negative definite twist are not necessarily hyperbolic. In the proof we show that in the neighborhood of a minimal periodic orbit of period n, the nth iterate of the map is again a twist map. This is true even though in general the composition of twist maps is not a twist map. © 1998 Elsevier Science B.V.

PACS: 03.20.+i; 05.45.+b

1. Introduction

We consider a discrete Lagrangian system on the configuration space Q, of dimension d. A discrete Lagrangian, $L(x, x'), x, x' \in Q$, is a generating function for a symplectic map $(x', y') = F(x, y)$ on $Q \times \mathbb{R}^d$, that is implicitly defined by (for a review, see Ref. [10])

$$y = -L_1(x, x'), \quad y' = L_2(x, x'). \tag{1}$$

The subscripts 1 and 2 denote the derivative with re...

assume that the (local) twist condition, $\det L_{12} \neq 0$, holds, so that x' can be determined, at least locally, as a function of (x, y). The dynamics can also be obtained from a variational principle: define the periodic action by

$$W_{mn} = \sum_{i=0}^{n-1} L(x_i, x_{i+1})|_{x_n = x_0 + m}. \tag{2}$$

When the configuration space is the torus, $Q = \mathbb{T}^d$, we can fix the period of the torus to 1 in every dimension and choose $m \in \mathbb{Z}^d$, otherwise we just set $m = 0$. It is easy to see that every critical point of W_{nm} corresponds to a periodic orbit of F with period n.

A minimal periodic orbit is a nondegenerate, local minimum of W_{nm} (we do not require it to be glob-ally minimizing...). A... Mather theory [10])...

expected to be important: for example every orbit on an invariant torus (that is a Lagrangian graph) is minimizing [9]. The purpose of this note is to establish the relation, if any, between the fact that the orbit is minimal and its stability type.

Relations between the index of a certain quadratic form (which is not the Hessian of the action) and the...

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stability type of fixed points of symplectic mappings
have been obtained in Ref. [1]. Similar results spe-

approach is different because we specifically look at
action minimizing orbits, i.e. the index of the Hessian
of the action is 0 (or maximal). In Ref. [2] it was
shown that there exist Lagrangian flows for which the

ing periodic orbits are not hyperbolic. This is very sim-

a completely different method. Although it is known

one flow [11], this requires one to treat time depen-
dent Lagrangian flows, which was not done in Ref.
[2].

with period n, and let $x_{i+1} = x_i$. The linearization
of the map at this orbit gives rise to an eigenvalue
problem with eigenvalues that we call μ, multipliers

a multiplier by

$$R = \frac{1}{4}\left(2 - \mu - \frac{1}{\mu}\right). \tag{3}$$

Since the multipliers for a symplectic map come in
reciprocal pairs μ and $1/\mu$, there are d residues in
dimension d, and their values completely determine
the stability type of the orbit. A multiplier is elliptic,
denoted "E" when $\mu = e^{i\phi}$ or equivalently when $0 \leqslant R \leqslant 1$. It is inverse hyperbolic, denoted "I", when
$1 < R$ and hyperbolic, denoted "H", when $R < 0$.

this latter case can occur only when $d \geqslant 2$.

With the notation

for the Hessian of L at (x_i, x_{i+1}) we can express the

[4]

$$DF(x_i, y_i) = \begin{pmatrix} -B_i^{-1}A_i & -B_i^{-1} \\ B_i^T - D_i B_i^{-1}A_i & -D_i B_i^{-1} \end{pmatrix}. \tag{5}$$

It is often more convenient to obtain the stability of
period n orbits directly from the Lagrangian formula-
tion. Using the abbreviation,

about the orbit is

$$B_{i-1}^T \delta x_{i-1} + P_i \delta x_i + B_i \delta x_{i+1} = 0. \tag{7}$$

system subject to the condition that $\delta x_{i+n} = \mu \delta x_i$.

period

$$M_1(\mu) = P_1 + \mu B_1 + \frac{1}{\mu} B_1^T \tag{8}$$

$$M_2(\mu) = \begin{pmatrix} P_1 & B_1 + B_2^T/\mu \\ B_1^T + \mu B_2 & P_2 \end{pmatrix}, \tag{9}$$

$$M_n(\mu) = \begin{pmatrix} P_1 & B_1 & & & \frac{1}{\mu}B_n^T \\ B_1^T & P_2 & B_2 & & \\ & & \ddots & & \\ & & B_{n-2}^T & P_{n-1} & B_{n-1} \\ \mu B_n & & & B_{n-1}^T & P_n \end{pmatrix},$$
$$n > 2. \tag{10}$$

The Hessian of the periodic action W_{mn} is given by
$M_n(1)$. the assumption that the periodic orbits under
consideration be minimal therefore is

$$M_n(1) > 0. \tag{11}$$

then the matrix $M_n(e^{i\phi})$ is Hermitian.

When $d = 1$, there is a simple relation between
the Hessian of the periodic action W_{mn} and the

$$\frac{1}{\det M_n(1)}$$

where $M_n(1) = D^2 W_{mn}$ is the Hessian and $B_i \equiv L_{12}(x_i, x_{i+1})$. For $d > 1$, there is no such simple rela-
tion, though the product of the residues can be written
similarly [7]. Eq. (12) implies that when $d = 1$ and
the twist is negative definite, nondegenerate minimal
orbits are hyperbolic. We will show that this is false

for $d > 1$: the multipliers of minimal periodic orbits can become elliptic.

In Section 2 we analyze minimal fixed points and establish the fact that they can be nonhyperbolic if the twist is either nonsymmetric or indefinite. For 4D maps we completely analyze the structure of minimizing fixed points in the case of symmetric ...

... minimal periodic orbits to that of a minimal fixed point. Clearly the period-n orbits of F are fixed points of the iterated map F^n. However, it is well known [10] that the iterate of a twist map is in general not a twist map. This does not preclude the possibility that the iterated map restricted to the neighborhood of a minimal periodic orbit is a twist map, which we will prove to be the case in Section 3. Finally we give two examples of minimizing periodic orbits which are elliptic.

2. Fixed points

... $P_1 + B_1 + B_1^T > 0$. Rewriting $M_1(\mu)$ to isolate this term gives

$$M_1(\mu) = M_1(1) + \frac{1}{2}\left(\mu + \frac{1}{\mu} - 2\right)(B_1 + B_1^T)$$
$$+ \frac{1}{2}\left(\mu - \frac{1}{\mu}\right)(B_1 - B_1^T). \tag{13}$$

For the physically interesting case when the twist B_1 is symmetric, the last term vanishes, and the spectrum is determined by the generalized eigenvalue problem

$$\det(M_1(1) - 4RB_1) = 0. \tag{14}$$

Since $M_1(1)$ is positive definite, and both matrices are symmetric, they can be simultaneously diagonalized. Thus the residue is obtained as the eigenvalue of ...

... out complex quadruplets of multipliers. Elliptic multipliers are possible for arbitrary symmetric B, and occur when $0 < R < 1$.

$$M_1(e^{i\phi}) = M_1(1) + 2(1 - \cos\phi)(-B_1). \tag{15}$$

This is positive definite since it is the sum of a positive definite matrix and a positive semidefinite matrix. Therefore, $\det M_1(\exp(i\phi)) \neq 0$, and there are no multipliers on the unit circle. Thus for negative definite twist a nondegenerate minimizing fixed point is hyperbolic in d dimensions.

If the twist B_1 is ... choosing $M_1(1)$ and B_1 diagonal.

If the twist B_1 is nonsymmetric the ... lem for R problem cannot be derived in this simple way. Introducing the symmetric part of the twist $\tilde{S} = (B_1 + B_1^T)/2$ and its antisymmetric part $\tilde{Y} = (B_1 - B_1^T)/2$ we can rewrite $\det(M_1(\mu)) = 0$ as

$$\det(M_1(1) - 4R\tilde{S} - 4\delta\tilde{Y}) = 0, \tag{16}$$

where $\delta = (1/\mu - \mu)/4 = \sqrt{R(1 - R)}$. By simultaneous diagonalization we can again simplify the problem in reducing $M_1(1)$ to the identity and $\tilde{S}$ to the diagonal S. Y denotes the transformed $\tilde{Y}$ which is still antisymmetric, and the ...

$$\det(1 - 4RS - 4\delta Y) = 0. \tag{17}$$

We know that this must be a polynomial in R, because the reflexivity of the characteristic polynomial for the multiplier μ [3] allows it to be rewritten as a polynomial of degree d in $\mu + 1/\mu$, or, equivalently, in R. To see this explicitly we employ the "cumulant expansion" for an arbitrary $n \times n$ matrix A,

$$\det(1 + \varepsilon A) = \sum_{i=0}^{n} \varepsilon^i Q_i(A), \tag{18}$$

where the cumulants (or up to a sign the coefficients of the characteristic polynomial of A) are recursively defined by

$$Q_0 = 1,$$

$$Q_i = \frac{1}{i}\sum_{k=1}^{i}(-1)^{k+1}Q_{i-k}(M)\operatorname{tr}A^k. \tag{19}$$

and eventually set $\epsilon = -1$. For large dimensions it is quite cumbersome to obtain explicit expressions for

the "characteristic polynomial" of R because in the expansion of $\operatorname{tr}\mathbf{A}^k$ we must compute terms of the form

$$\operatorname{tr}(R\mathbf{S} + \delta\mathbf{Y})^k = \sum_{j+l=k} \rho^j \delta^l \operatorname{tr}(\sigma(\mathbf{S}, j, \mathbf{Y}, l)), \qquad (20)$$

mutative!) products with j factors $\mathbf{S}$ and l factors $\mathbf{Y}$ in all possible orderings. Since we can cyclically permute under the trace a lot of terms can be combined.

contain traces of products of $\mathbf{S}$ and $\mathbf{Y}$ for $k > 2$. For $d = 2, 3$, we obtain

$$0 = \det(\mathbf{1} - 4R\mathbf{S}) - 8R(1 - R)\operatorname{tr}(\mathbf{Y}^2), \qquad (21)$$

$$0 = \det(\mathbf{1} - 4R\mathbf{S}) - 8R(1 - R)$$
$$\times (\operatorname{tr}(\mathbf{Y}^2)(1 - 4R\operatorname{tr}\mathbf{S}) + 8R\operatorname{tr}(\mathbf{S}\mathbf{Y}^2)) = 0. \qquad (22)$$

number of $\mathbf{Y}$ in the sequence of $\mathbf{S}$ and $\mathbf{Y}$. If reading the sequence backwards is the same sequence, then this term is antisymmetric and its trace vanishes. If read-

sum is antisymmetric, hence vanishes under the trace.

R of degree d.

If $\mu = 1$ then $\delta = 0$ and $R = 0$ such that the general determinant (17) can never vanish. This means that a minimizing orbit cannot undergo a saddle node bifurcation (without losing the minimizing property).

$$+ \det\mathbf{M} - 4R(1 - R)\operatorname{tr}(\tilde{\mathbf{Y}}^2) \qquad (23)$$

$$= (4d_1 R - 1)(4d_2 R - 1) + 16a^2 R(R - 1). \qquad (24)$$

minimizing) fixed points of natural maps is obtained.

sential parameters $\mathbf{S} = \operatorname{diag}(d_1, d_2)$ and a, the single entry of the antisymmetric $\mathbf{Y}$. The polynomial determining ρ is given by (17) respectively (21), or more explicitly,
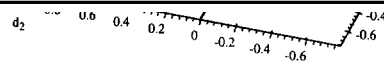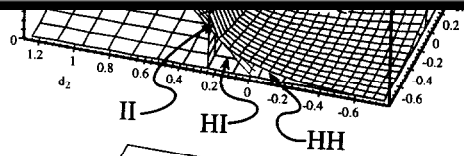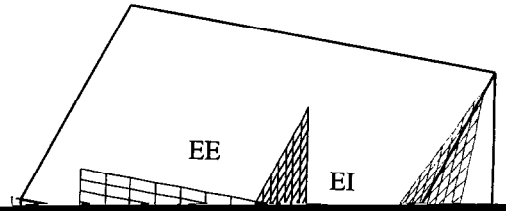
(d_1, d_2) is therefore divided into 9 regions by the 4 corresponds to multipliers of type HH. The transition from HH to any region in the adjacent quadrants is not a regular bifurcation, because it induces R to pass through infinity. In a smooth system this is impossible.

Basically it means that the signature of the symmetric twist is preserved under smooth parameter variation, which is by definition true in the general case. If the signature of the twist is mixed, we have *HE* or *HI*, and if it is positive, then we have *II*, *IE* or *EE*, the transitions taking place at $d_i = 1/4$. Now making a

volve driving R through infinity, i.e. making the twist singular.

The main change occurs above the *HH* and *II* region. Increasing a leads to a complex bifurcation where four real multipliers collide and turn complex, entering the region *CQ*. Increasing a further leads to the inverse complex bifurcation in which four complex multipliers collide on the unit circle, hence creating four elliptic multipliers. Since these elliptic multipliers are created in a complex bifurcation their Krein signatures must be different. Note that even though it looks like the two *EH* regions are disconnected, this is due to the ambiguity in the ordering of the eigenvalues in **S**. In the full parameter space they are connected and together with *IH* form a region bounded by $\det \mathbf{M}_1(1) = 0$. All the other regions are smoothly connected; only for symmetric twist the *HH* region is separated from the others.

part can turn the minimizing hyperbolic fixed point elliptic via an (inverse) Krein collision.

3. Periodic orbits

We now turn to the calculation of stability of pe-

connection with Schur's complement [12] of $\mathbf{M}_2$, in

twist generating function of a minimal fixed point. For $n > 2$ we will directly work with Schur's complement to establish this result. Recall that the Schur complement $(\mathbf{M} \mid \mathbf{D})$ of $\mathbf{M}$ with respect to $\mathbf{D}$ is defined by the following factorization,

such that

$$(\mathbf{M} \mid \mathbf{D}) = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} . \tag{26}$$

A and **D** are square matrices; if they have different dimensions then **B** and **C** are not square matrices. The

$$\det \mathbf{M} = \det(\mathbf{M} \mid \mathbf{D}) \det \mathbf{D} . \tag{27}$$

We will need the fact [12] that the Schur complement of a symmetric positive definite matrix is symmetric and positive definite. This is easily seen because transforming the quadratic form corresponding to **M** with

$$\mathbf{T} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{B}^t & \mathbf{1} \end{pmatrix}$$

gives $\mathbf{T}^t\mathbf{M}\mathbf{T} = \mathrm{diag}((\mathbf{M} \mid \mathbf{D}), \mathbf{D})$. $\tag{28}$

For a periodic orbit of period $n = 2$ we could multiply $DF(x_2, y_2)$ and $DF(x_1, y_1)$, and identify the resulting matrix to be of the form (5). It is simpler to consider the second difference equation for the period of 2 orbit,

$$-\frac{1}{\mu}\mathbf{B}_2^t \delta x_2 + \mathbf{P}_1 \delta x_1 + \mathbf{B}_1 \delta x_2 = 0 . \tag{30}$$

Solving the first equation for δx_2 and eliminating it in the second directly gives $\mathbf{M}_1^{(2)}$. The superscript 2 denotes that the matrix is that of a fixed point corresponding to a period 2 orbit. By comparison with (8), we find

$$\mathbf{P}_1^{(2)} = \mathbf{P}_1 - \mathbf{B}_2^T \mathbf{P}_2^{-1} \mathbf{B}_2 - \mathbf{B}_1 \mathbf{P}_2^{-1} \mathbf{B}_1^T . \tag{32}$$

$\mathbf{B}_1^{(2)}$ and $\mathbf{P}_1^{(2)} = \mathbf{A}_1^{(2)} + \mathbf{D}_1^{(2)}$ define a generating function by (4) for the iterated map. The splitting of $\mathbf{P}_1^{(2)}$ into $\mathbf{A}_1^{(2)}$ and $\mathbf{D}_1^{(2)}$ is arbitrary for our purposes; only $\mathbf{P}_1^{(2)}$ enters the stability formulae.

Our task is to show that the fact that the periodic orbit is minimal, $\mathbf{M}_n(1) > 0$ implies that the new twist

is implied by $\mathbf{M}_2(1) > 0$, because $\mathbf{P}_2$ is a principal subblock[2] of $\mathbf{M}_2$.

To show that $\mathbf{M}_1^{(2)}(1) > 0$, we note that

$$\mathbf{M}_1^{(2)}(1) = \mathbf{P}_1^{(2)} + \mathbf{B}_1^{(2)} + (\mathbf{B}_1^2)^{\mathsf{T}} \tag{33}$$

$$= \mathbf{P}_1 - (\mathbf{B}_1 + \mathbf{B}_2^{\mathsf{T}})\mathbf{P}_2^{-1}(\mathbf{B}_1^{\mathsf{T}} + \mathbf{B}_2) \tag{34}$$

$$= (\mathbf{M}_2(1) \mid \mathbf{P}_2) . \tag{35}$$

Now the desired statement immediately follows because the Schur complement of a symmetric positive definite matrix is again symmetric positive definite.

case of period n we directly use Schur's complement on the matrix $\mathbf{M}_n(\mu)$ to recursively reduce dimension by d in each step. The final result after $n-1$ steps is $\mathbf{M}^{(n)}(\mu)$ where the superscript is an iteration index.

From this we can identify the twist $\mathbf{B}_1^{(n)}$ of the generating function for the orbit ... of the map in the neighborhood of the minimal period n orbit via (8). The proof proceeds by induction. The initial matrix gets the iteration index 1, $\mathbf{M}_n(\mu) = \mathbf{M}_n^{(1)}(\mu)$. The iteration rule is

$$\mathbf{M}_{k-1}^{(i+1)}(\mu) = (\mathbf{M}_k^{(i)}(\mu) \mid \mathbf{P}_k^i) , \tag{36}$$

or more explicitly,

$$\mathbf{P}^{(i+1)} = \mathbf{P}^{(i)} - \mathbf{B}^{(i)}(\mathbf{P}^{(i)})^{-1}(\mathbf{B}^{(i)})^{\mathsf{T}} \tag{38}$$

$$\mathbf{B}_{k-1}^{(i+1)} = -\mathbf{B}_{k-1}^{(i)}(\mathbf{P}_k^{(i)})^{-1}\mathbf{B}_k^{(i)} , \tag{39}$$

$$\mathbf{B}_j^{(i+1)} = \mathbf{B}_j^{(i)} , \quad j = 1, \ldots, k-2 , \tag{40}$$

$$\mathbf{P}^{(i+1)} = \mathbf{P}^{(i)} \quad \ldots \tag{41}$$

The last two lines merely state that these entries do not ...

... in reducing the dimension by d. Note that for $k = 2$ these formulas collapse to (31). Parts of this iteration formula are identical to those reported in Refs. [9] and [6]. The formulation we have chosen here allows ...

to fixed points of twist maps. This fact has not been realized before, and we are now going to prove it.

Since we start a positive definite matrix $\mathbf{M}_n^{(1)}(1) > 0$, the next iterate constructed by Schur's complement

[2] By principal subblock we mean a block that is centered on the diagonal.

is also positive definite. By induction all $\mathbf{M}_{n-i}^{(i+1)}(1) > 0$. By assumption $\mathbf{B}_n^{(1)}$ and $\mathbf{B}_{n-1}^{(1)}$ in $\mathbf{M}_n^{(1)}$ are nonsingular, and since $\mathbf{P}_n^{(1)}$ is a principal subblock of the positive definite matrix $\mathbf{M}_n^{(1)}$ it is positive definite, and therefore also nonsingular. In the iteration step from i to $i+1$, $k = n-i+1$, one of the relevant twist matrices is not changed, $\mathbf{B}_{k-2}^{(i+1)} = \mathbf{B}_{k-2}^{(i)}$, the other one obtained from (39) is also nonsingular because by assumption (1) the two matrices on the right-hand side of (39) are nonsingular, and (2) the matrix $\mathbf{P}_k^{(i)}$ in the same equation is nonsingular because it is a principal sub...

have shown that the twist stays nonsingular and that the matrices $\mathbf{M}^{(i+1)}(1)$ stay positive definite.

Although in general the composition of twist maps does not give a twist map we have shown that in the neighborhood of a minimal period n orbit there exists a local generating function with nonsingular twist for the n times iterated map. The essential observation concerning stability of minimal periodic orbits is ...

... nite twist is *not* stable under this iteration. The final ...

$$\mathbf{B}_1^{(n)} = \mathbf{B}_1 \prod_{i=2}^{n} (\mathbf{P}_i^{(n-i+1)})^{-1}(-\mathbf{B}_i) . \tag{42}$$

maps $L(x, x') = (x' - x)^2/2 - U(x)$ which have ...

for $n > 2$ we obtain the product of $n-1$ symmetric positive definite matrices which is in general neither symmetric nor positive definite. However, if the matrices $\mathbf{P}^{(n-i+1)}$ commute with each other then their product is symmetric and positive definite. This can ...

therefore commute. But this is true only if the potential separates, such that we are back to the case $d = 1$.

Note that if we apply the determinant formula for we obtain

$$\det \mathbf{M}_{k-1}^{(i+1)}(\mu) = \det(\mathbf{M}_k^{(i)}(\mu) \mid \mathbf{P}_k^{(i)}) \det \mathbf{P}_k^{(i)} . \tag{43}$$

In each step the last factor is nonzero, such that we can ignore all of them and find

$$0 = \det \mathbf{M}_n^{(1)}(\mu) \iff 0 = \det \mathbf{M}_1^{(n)}(\mu) , \tag{44}$$

Finally we give two examples of minimizing periodic orbits that are not hyperbolic. The first exam-

$$P_1 = \text{diag}(5/3, 3/4), \quad P_2 = \text{diag}(2/3, 3/2) \qquad (45)$$

and

$$B_1 = \begin{pmatrix} -1/2 & b \\ b & -1/2 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -1/2 & -b \end{pmatrix} \qquad (46)$$

which are symmetric negative definite twist matrices as long as $|b| < 1/2$. The resulting matrix $M_2(1)$ is positive definite. However, the resulting multipliers

of $M_2(1)$ are given by

$$(9\lambda^2 - 21\lambda + 1)(8\lambda^2 - 18\lambda + 1) = 0, \qquad (47)$$

which are all positive, while the multipliers are given by

ues for $B_i = \text{diag}(-1, -1)$, we have to go to period

$$P_3 = \begin{pmatrix} 7.2 & 2 \\ 2 & 3.8 \end{pmatrix}. \qquad (49)$$

Since the eigenvalues of some P_i have to be quite dif-

be elliptic. This result was obtained in three stages. First we showed that a nondegenerate fixed point with minimal action of a twist map with negative definite symmetric twist is hyperbolic in any number of dimensions. For application to the case of periodic orbits we noted that if the twist is not definite or not symmet-

4. Discussion

fixed points that are not hyperbolic, provided the Lagrangian is bounded from below. Our main point was to show that there exist maps whose minimal fixed

In order to show this, we first derived the interesting result that in the neighborhood of a minimal periodic orbit the iteration of a twist map is again a twist map, which is not always true globally. The key to this observation was the use of Schur's complement to recursively reduce the dimension of the Hessian of the pe-

ist autonomous Lagrangian flows for which the minimizing orbits are not hyperbolic. Our result for maps

it amounts to treating Lagrangians with explicit time dependence

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