Physica D 71 (1994) 1–17
North-Holland

SSDI: 0167-2789(93)E0246-8

PHYSICA D

[redacted]

J.L. Tennyson[a,1], J.D. Meiss[b] and P.J. Morrison[c]

[a] *Stanford Linear Accelerator Center, Stanford University, Palo Alto, CA 94305, USA*
[b] *Program in Applied Mathematics, Box 526, University of Colorado, Boulder, CO 80309, USA*
[c] *Department of Physics and*
[redacted]

[redacted]
Revised manuscript received 12 April 1993
Accepted 14 April 1993
[redacted]

The effect of self-consistency on Hamiltonian systems with a large number of degrees of freedom is investigated [redacted] is observed that the system relaxes into a time asymptotic periodic state where only a few collective degrees are active; [redacted] low degree-of-freedom model is derived that treats the clump as a single *macroparticle*, interacting with the wave and chaotic sea. The uniform chaotic sea is modeled by a fluid waterbag, where the waterbag boundaries correspond [redacted]

[redacted]

Chaotic motion in Hamiltonian systems is [redacted] gree of freedom [1]. Often, the systems studied are low dimensional approximations of many degree-of-freedom systems. In some cases, such [redacted] tions can be given with only a few degrees of [redacted] such as galactic dynamics, where the number of degrees of freedom is essentially infinite. Generally, one expects such systems to exhibit greater chaos when the dimension increases; [redacted] of degrees of freedom. Even in high dimensional [redacted] sional approximation, for example to study the motion of a single star in a given galactic gravitational potential—this was the motivation for [redacted] erences in [1]). Such an approximation is not [redacted] this type include the motion of charged particles in electromagnetic fields, where the fields produced by the particles are ignored; the motion of tracer particles in a fluid, where the influence of these particles on the fluid velocity field is ignored (the passive advection problem); and

---

[1] Posthumous. Prepared by J.D.M. and P.J.M.

the so-called “kinematic” dynamo, where a ve- [redacted]

There has been little work on the effect of self- [redacted] sible in a system with a large number of degrees [redacted] to result in dynamics with “effectively” few degrees of freedom.

[redacted]

Hamiltonian for each particle has one and a half degrees of freedom, and so the motion can be chaotic. However, each of the parti- [redacted] of the field is given. Thus each particle experi- [redacted] extent that the other particles contribute to the single mode of the field. This is in contrast to [redacted] particle. This latter case is considerably more difficult.

Models similar to the one described above may be appropriate for many physical situations; for example, a galaxy with a predominantly axisymmetric gravitational potential that is per- [redacted] possibly chaotic motion in the corresponding field. Similar effects occur for planetary rings, [redacted] O'Neil, Winfrey and Malmberg (hereafter re- [redacted] beam of charged particles moves in a background neutral plasma. The system is unstable to the formation of electrostatic plasma waves. [redacted] remainder of the modes—this is easily justified [redacted] showed that the wave grows in amplitude un- [redacted] and begins to oscillate in amplitude as the beam particles slosh in the wave potential. At this [redacted]

We neglect these modes; this is justified, for example, if the system has a finite length, and the sideband wavenumbers are forbidden by [redacted] a rigid bar in phase space. When the beam is [redacted] tate. Mynick and Kaufman computed the frequency shift and amplitude oscillations of the [redacted] of the plasma wave oscillates, the beam particles can experience chaotic motion. They studied the motion of a test particle in a model of this oscillating field and showed that much of the test particle phase space is indeed chaotic. However, there is an island in the phase space where [redacted] the correct region of phase space to be trapped in this oscillating island. Later Adam, Laval and [redacted] this two degree-of-freedom system is integrable [redacted] it was shown that the macroparticle system has solutions which correspond to periodic oscillations of the bunch in the wave.

Related self-consistent problems include the interaction of a single particle with many waves [8] and the interaction of one wave with many other waves [9]. The more complicated

[redacted]

wave–particle turbulence, and it is not clear if the analysis of this paper can give any insight into this case.

In Section 2 we review the derivation of the OWM model, obtaining the Hamiltonian formu-

[redacted]

equations. Section 3 discusses numerical solutions of the OWM equations with up to 10^5 beam

[redacted]

served at least 100 periods of these oscillations; as far as we can determine, the oscillations per-

[redacted]

ticle in this periodic potential, showing that a substantial portion of the original beam is indeed trapped in a stable island in the test par-

[redacted]

beam finds itself in the chaotic region of phase space, and spreads more or less uniformly over this region. The upper and lower boundaries of this "chaotic sea" are formed from invariant tori of the test particle system.

In Section 4 we construct a four degree-of-

[redacted]

freedom model. One degree of freedom is the wave, the second corresponds to the trapped

[redacted]

tions of the boundary of the chaotic sea and are derived from the "waterbag" approximation.

[redacted]

case the simulations show that the phase space density of the chaotic particles is indeed nearly constant and the boundaries of the chaotic zone are formed from invariant surfaces well outside the oscillating separatrix of the wave. We model these boundaries with sinusoidal curves, an assumption consistent with that of the single mode in the potential. Finally, the frequency shift of the trapped particle oscillations due to

[redacted]

2. Single wave model

O'Neil, Winfrey, and Malmberg (OWM) [2]

[redacted]

growth and saturation of the weak beam–plasma instability. In this section we briefly review the

[redacted]

sented by Mynick and Kaufman [5], discuss linear instability, and finally consider a special case

[redacted]

2.1. Derivation

To obtain the single wave model, the response

[redacted]

separately. We consider only the one dimensional, collisionless, nonrelativistic, electrostatic case. The total electron density

$$n(x,t) = n_p(x,t) + n_b(x,t)$$

[redacted]

[redacted]

$$m\ddot{x}_j = -eE(x_j,t)\,, \qquad (2)$$

[redacted]

phase velocity of the resulting instability is much larger than the velocities of particles in the background plasma: the plasma responds nonresonantly, and trapping effects of plasma particles in the wave can be neglected. This implies that [illegible]

$$4\pi e n_{\mathrm{p}}(x,t) = (1 - \hat{\epsilon})\varphi''(x,t)\,, \qquad (3)$$

where φ is the electrostatic potential, $E = -\varphi'$. Substituting this into Poisson's equation for φ [illegible]

$$\hat{\epsilon}\varphi = 4\pi e n_{\mathrm{b}}(x,t)\,. \qquad (4)$$

[illegible] tion is most easily treated by Fourier transform. In this representation the dielectric operator becomes a real function, $\epsilon(k,\omega)$. For a weak beam [illegible] that the electrostatic response is dominated by [illegible] is a reasonable approximation to expand ϵ about one such zero retaining only the first derivative of ϵ with respect to ω:

$$\begin{aligned}\epsilon(k,\omega) &\approx \epsilon(k,\omega_0) + \left.\frac{\partial\epsilon}{\partial\omega}\right|_{\omega=\omega_0}(\omega-\omega_0)\\ &= \epsilon'(\omega-\omega_0)\,. \qquad (5)\end{aligned}$$

For example, for a cold plasma $\epsilon = 1 - \omega_{\mathrm{p}}^2/\omega^2$, and $\partial\epsilon/\partial\omega|_{\omega=\omega_0} \equiv \epsilon' = 2/\omega_{\mathrm{p}}$. Transforming back to the time domain and using Eq. (4) then gives

$$\dot{E}_k + \mathrm{i}\omega_0 E_k = \frac{4\pi e}{kL\epsilon'}\sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}kx_j(t)}\,, \qquad (6)$$

where we have used the Fourier transform of the beam density of Eq. (1):

[illegible]

$$= \frac{1}{L}\sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}kx_j(t)}\,, \qquad (7)$$

[illegible]

At this point we assume that the electrostatic [illegible] k-space is relatively narrow in units of $2\pi/L$. In this case, if k represents the most unstable mode, the amplitude of all other Fourier com- [illegible] single wave during the linear growth stage. Of course, some time after nonlinear saturation of [illegible] stable spectrum depends on the small parameter $(n_{\mathrm{b}}/n_{\mathrm{p}})^{1/3}$, so that the single wave model will be most appropriate in the weak beam case.

[illegible] yields

[illegible]

$$\dot{p}_j = -e\left(E_k\,\mathrm{e}^{\mathrm{i}kx_j} + E_{-k}\,\mathrm{e}^{-\mathrm{i}kx_j}\right)\,. \qquad (8)$$

Equation (8) together with Eq. (6) are the closed dynamical system that governs the interaction of a single wave with the beam particles.

2.2. Hamiltonian structure and derivation

Now consider the derivation of the equations of motion, Eqs. (6) and (8), within the Hamiltonian context. The derivation proceeds by first considering the kinematics, i.e. the dynamical variables used to describe the state of the system, and then the dynamics, obtained by finding the appropriate Hamiltonian.

We begin the first part by supposing that the electrons are described by specifying their [illegible] $1,2,\ldots,M$. The first $N(<M)$ of these particles are singled out to represent the beam dynamics, while the remaining $M-N$ particles represent the background plasma. The phase space density, i.e. distribution function, can be [illegible]

$$f(x,p,t) = f_b(x,p,t) + f_p(x,p,t)$$

[redacted]

$$+ \sum^{M} \delta(x - x_j(t))\delta(p - p_j(t)) \quad (9)$$

[redacted]

The Poisson bracket in terms of (x_j,p_j) where $j = 1,2,\ldots,M$, has the standard canonical form

[redacted]

$$[g,h] = \sum_{j=1} \left(\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial p_j} - \frac{\partial h}{\partial x_j}\frac{\partial g}{\partial p_j}\right)$$
$$\equiv [g,h]_b + [g,h]_p, \quad (10)$$

where g and h are functions defined on phase space.

It is desired to describe the state of system

[redacted]

for the beam particles. However, for the background plasma, the phase space coordinates of these particles will be replaced by a Vlasov type distribution function, f_p. This can be achieved by mapping the Poisson bracket of Eq. (10) to

[redacted]

is a *noncanonical* variable, therefore the resulting Poisson bracket possesses noncanonical form [11]. In order to effect this transformation the chain rule [12,13] for functional derivatives is required. Suppose

$$g(x_j,p_j) = G[f_p], \quad (11)$$

where $j = N+1, N+2, \ldots, M$. Here $G[f_p]$ is a *functional* of f_p; the relationship between the phase space function g and the functional G is

[redacted]

is obtained by varying both sides of this equation:

$$\delta g = \sum_{j=N+1}^{M} \left(\frac{\partial g}{\partial x_j}\delta x_j + \frac{\partial g}{\partial p_j}\delta p_j\right)$$
$$= \delta G$$

$$= \int \frac{\delta G}{\delta f_p} \sum_{j=N+1}^{M} \left(\frac{\delta f_p}{\delta x_j}\delta x_j + \frac{\delta f_p}{\delta p_j}\delta p_j\right) \mathrm{d}x\,\mathrm{d}p$$

[redacted]

The operators $\delta f/\delta x$ and $\delta f/\delta p$ are obtained by variation of Eq. (9) with respect to the plasma

[redacted]

result is finally

$$\frac{\partial g}{\partial x_j} = \int \left(\frac{\partial}{\partial x}\frac{\delta G}{\delta f_p}\right) \delta(x - x_j)\,\delta(p - p_j)\,\mathrm{d}x\,\mathrm{d}p$$
$$= \left(\frac{\partial}{\partial x}\frac{\delta G}{\delta f_p}\right)\Big|_{(x_j,p_j)}$$
$$\frac{\partial g}{\partial p_j} = \int \left(\frac{\partial}{\partial p}\frac{\delta G}{\delta f_p}\right) \delta(x - x_j)\,\delta(p - p_j)\,\mathrm{d}x\,\mathrm{d}p$$

[redacted]

$$= \left(\frac{\partial}{\partial p}\frac{\delta G}{\delta f_p}\right)\Big|_{(x_j,p_j)}. \quad (13)$$

Insertion of Eq. (13) into the second term of Eq. (10) yields the bracket

[redacted]

$$+ \sum_{j=1} \left(\frac{\partial G}{\partial x_j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial x_j}\frac{\partial G}{\partial p_j}\right), \quad (14)$$

where

$$\{g,h\} \equiv \frac{\partial g}{\partial x}\frac{\partial h}{\partial p} - \frac{\partial h}{\partial x}\frac{\partial g}{\partial p}. \quad (15)$$

Here the quantities G and H are functionals of f_p, but according to Eq. (9) they can be thought of as ordinary functions of the beam particle coordinates (x_j,p_j) where $j = 1,2,\ldots,N$. Note that discreteness has now disappeared from f_p.

[redacted]

ground Vlasov plasma electrons is obtained by inserting

$$f(x,p,t) = f_p(x,p,t) + \sum_{j=1}^{N} \delta(x - x_j(t))\delta(p - p_j(t)) \quad (16)$$

[redacted]

$$H[f_{\mathrm{p}}; x_j, p_j] = \frac{1}{2m}\int p^2 f_{\mathrm{p}}\,\mathrm{d}x\,\mathrm{d}p - \frac{e}{2}\int f_{\mathrm{p}}\varphi_{\mathrm{p}}\,\mathrm{d}x + \sum_{j=1}^{N}\left(\frac{p_j^2}{2m} - e\varphi_{\mathrm{p}}(x_j) - \tfrac{1}{2}e\varphi_{\mathrm{b}}(x_j)\right), \quad (18)$$

[redacted]

where $\varphi_{\mathrm{p}}(x_j)$ and $\varphi_{\mathrm{b}}(x_j)$ are the contributions to the electrostatic potential of the plasma and

[redacted]

yields the hybrid system.

Now we can turn to the task of obtaining, from the hybrid system, the approximate system of

[redacted]

sumed to be described by an equilibrium distribution function of compact support in velocity, plus the single linear wave, whose phase velocity

[redacted]

wave–particle effects are eliminated in the back-

[redacted]

of [14] and [15] implies that the linearization of the plasma energy becomes identically the well-known expression for the dielectric energy of a plasma wave. Second, the self-interaction potential of the beam, φ_{b}, is neglected in comparison to that of the plasma, φ_{p}, a justifiable assumption in light of smallness of $n_{\mathrm{b}}/n_{\mathrm{p}}$. Thus, Eq. (18) becomes

$$H(E_k, E_{-k}, x_j, p_j) = \frac{L}{4\pi}\,\omega_0\,\epsilon'|E_k|^2 + \sum_{j=1}^{N}\left(\frac{p_j^2}{2m} - \frac{\mathrm{i}\,e}{k}E_k\,\mathrm{e}^{\mathrm{i}kx_j} + \frac{\mathrm{i}\,e}{k}E_{-k}\,\mathrm{e}^{-\mathrm{i}kx_j}\right). \quad (19)$$

It remains to find the appropriate Poisson bracket in terms of E_k and E_{-k} instead of f_{p}. Since the plasma is in essence being modeled as a fluid, an easy way to obtain this is to map

[redacted]

$$[g,h] = \sum_{j=1}^{N}\left(\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial p_j} - \frac{\partial h}{\partial x_j}\frac{\partial g}{\partial p_j}\right) - \frac{\mathrm{i}}{L}\frac{4\pi}{\epsilon'}\left(\frac{\partial g}{\partial E_k}\frac{\partial h}{\partial E_{-k}} - \frac{\partial h}{\partial E_k}\frac{\partial g}{\partial E_{-k}}\right). \quad (20)$$

[redacted]

Eqs. (6) and (8) in the form

[redacted]

The bracket of Eq. (20) is not quite canonical; however, with the substitution

[redacted]

$$E_{-k} = \mathrm{i}\left(\frac{4\pi}{L\epsilon'}\right)^{1/2}\mathcal{J}^{1/2}\,\mathrm{e}^{\mathrm{i}\vartheta}, \quad (22)$$

the electric field is expressed in terms of conven-

[redacted]

$$[g,h] = \sum_{j=1}^{N}\left(\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial p_j} - \frac{\partial h}{\partial x_j}\frac{\partial g}{\partial p_j}\right) + \left(\frac{\partial g}{\partial \vartheta}\frac{\partial h}{\partial \mathcal{J}} - \frac{\partial h}{\partial \vartheta}\frac{\partial g}{\partial \mathcal{J}}\right), \quad (23)$$

while the Hamiltonian of Eq. (19) becomes

$$H(\vartheta, \mathcal{J}, x_j, p_j) = \omega_0\mathcal{J} + \sum_{j=1}^{N}\left[\frac{p_j^2}{2m} - \frac{2e}{k}\left(\frac{4\pi}{L\epsilon'}\right)^{1/2}\mathcal{J}^{1/2}\cos(kx_j - \vartheta)\right]. \quad (24)$$

To complete the derivation, it is convenient to introduce scaled, dimensionless variables based on the fundamental frequency,

$$\omega_{\mathrm{b}}^3 = \frac{4\pi\,\mathrm{e}^2 N}{mL\epsilon'}. \quad (25)$$

Here ω_{b} is a harmonic mean of the beam's plasma frequency and $1/\epsilon'$, which is of order

[redacted]

small parameter $(n_b/n_p)^{1/3}$ is represented by

[redacted]

$$H(J,\theta,p_j,\xi_j) = \sum_{j=1}^{N}\left[\tfrac{1}{2}p_j^2 - 2\left(\frac{J}{N}\right)^{1/2}\cos(\xi_j-\theta)\right], \qquad (26)$$

where the dimensionless variables are defined by

[redacted]

frame moving at the phase velocity ω_0/k. This

[redacted]

It is often convenient to use a noncanonical wave amplitude variable instead of action–angle variables. This is easily done if we use as inde-

[redacted]

plex conjugate Φ^* defined by

$$\Phi(\tau) = \left(\frac{J}{N}\right)^{1/2} e^{-i\theta}. \qquad (28)$$

In these coordinates the Poisson bracket be-

[redacted]

and the Hamiltonian takes the form

$$H = \sum_{j=1}^{N}\left(\tfrac{1}{2}p_j^2 - \Phi\, e^{i\xi_j} - \Phi^*\, e^{-i\xi_j}\right), \qquad (30)$$

[redacted]

the first term is the particle kinetic energy and the last two represent the electrostatic potential energy. The equations of motion are obtained from the Poisson bracket,

[redacted]

$$\frac{}{d\tau} = [\Phi,H] = [\Phi,\Phi^*]\frac{}{\partial\Phi^*}, \qquad (31)$$

[redacted]

$$\frac{d\Phi}{d\tau} = \frac{i}{N}\sum_{j=1}^{N} e^{-i\xi_j},$$

$$\frac{d^2\xi_j}{d\tau^2} = i\Phi\, e^{i\xi_j} - i\Phi^*\, e^{-i\xi_j}. \qquad (32)$$

Note that these equations hold for arbitrary choices of the physical parameters, such as e/m

[redacted]

ship between scaled variables and physical variables which changes. (Of course, the single wave

[redacted]

symmetries) which is the translation, $\xi_j \to \xi_j +$

[redacted]

$$P \equiv \sum_{j=1} p_j + J \qquad (33)$$

[redacted]

$P\theta + \sum p'_j(\xi_j - \theta)$ which gives the new momenta, $(p'_j = p_j, P)$, and angles, $(\psi_j = \xi_j - \theta, \theta)$. The new Hamiltonian is

[redacted]

$$= \sum_{j=1}\left[\tfrac{1}{2}p_j^2 - \frac{}{\sqrt{N}}\left(P - \sum_{k=1} p_k\right) \cos\psi_j\right], \qquad (34)$$

which has effectively N degrees of freedom since

[redacted]

2.3. Linear instability

To establish the fact that the Hamiltonian of Eq. (30) properly describes at least the linear stage of the weak beam–plasma instability, con

[redacted]

0. Linearizing the Hamiltonian about this equi-

[redacted]

(35)

The resulting linear equations of motion can be straightforwardly diagonalized to obtain the characteristic polynomial $\omega^{2N-4}(\omega^6 - 1) = 0$, where the solutions have been assumed to vary as

[redacted]

required because of the Hamiltonian nature of the flow (recall that if ω is an eigenvalue then ω^*, $-\omega$ and $-\omega^*$ must also be eigenvalues). In dimensional units, using Eq. (27), we have

$$\hat{\omega}_j = \omega_b \, e^{ij\pi/3} , \quad j = 0, 1, \ldots, 5 , \qquad (36)$$

which includes the unstable beam–plasma mode (the case $j = 2$). We can physically identify the eigenmodes by considering the equations of motion. Differentiating the equation for Φ twice and substituting for $\ddot{\xi}$ gives

[redacted]

$$\frac{d^3 \delta\Phi}{d\tau^3} = \frac{i}{N} \sum_{j=1}^{N} e^{-i\xi_j} \delta\xi_j = i\delta\Phi , \qquad (37)$$

upon noting that $\sum e^{-2i\xi_j^0} = 0$. This shows that the nonzero frequencies are associated with nonzero Φ. The eigenmodes for the conjugate roots, ω^*, $-\omega$ and $-\omega^*$, are the same as that for ω except for varying choices of signs. The remaining roots of the characteristic equation ($\omega = 0$ of multiplicity $2N-4$) have eigenmodes

[redacted]

$N-2$ independent solutions of $\sum e^{-i\xi_j} \delta\xi_j = 0$. The double multiplicity of each of these roots arises from allowing the moments to be nonzero

[redacted]

As will be seen in the next section, the linear

[redacted]

the particles in the electrostatic well of the wave.

[redacted]

The only exactly soluble case of the Hamiltonian Eq. (26) appears to be that of a single par-

[redacted]

tum, Eq. (33), reduces this case to one effective degree of freedom, and it can be integrated by quadrature. When N is larger than unity we expect to lose integrability. It will be of interest later to consider the case of N_m particles clumped

[redacted]

$$H = \frac{p^2}{2N_m} - 2N_m \left(\frac{J}{N}\right)^{1/2} \cos(\xi - \theta) , \qquad (38)$$

where $p = N_m p_1 = N_m p_2 \ldots$ is the macroparticle momentum. The Hamiltonian H can be reduced to one degree of freedom by defining the total momentum $P = p + J$ as before to obtain

$$H = \frac{p^2}{2N_m} - 2N_m \left(\frac{P - p}{N}\right)^{1/2} \cos\psi . \qquad (39)$$

The equations for this case were studied in detail by Adam, Laval and Mendonca [7], who did not use the Hamiltonian approach.

The Hamiltonian Eq. (39) has in general three

[redacted]

nondegenerate fixed points. These occur at the points defined by

$$p_0^3 - p_0^2 P + N_m{}^3 \frac{N_m}{N} = 0 , \quad \psi_0 = 0 \text{ or } \pi . \qquad (40)$$

The fixed point with $(p_0 < 0, \psi_0 = 0)$ is stable and corresponds to the macroparticle sitting in the bottom of the potential well. The two fixed points with $(p_0 > 0, \psi = \pi)$ are less intuitive. These exist only if $P > 3N_m (N_m/4N)^{1/3}$. They

[redacted]

potential well. The lower momentum particle is unstable, while the larger momentum particle is actually stable. This can be seen in Fig. 1, which

[redacted]

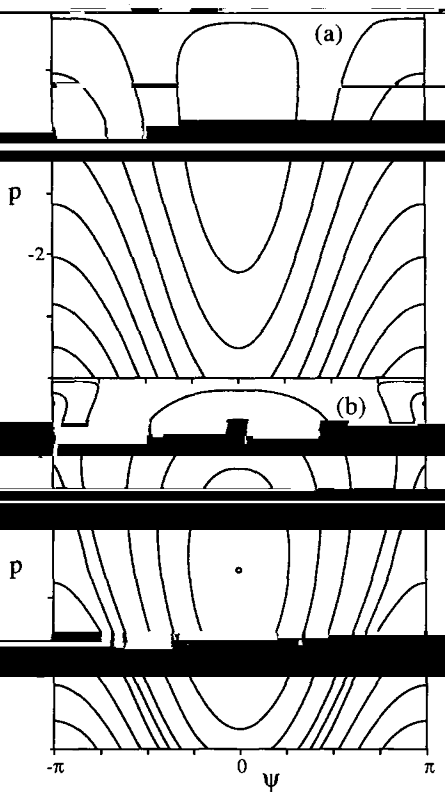

Fig. 1. Contours of H, Eq. (39), in the single particle phase space (p, ψ). Units of p (and P) are $N_{\mathrm{m}}\,(N_{\mathrm{m}}/N)^{1/3}$. In the upper figure $P = 0$ and there is one fixed point; in the lower figure $P = 2$ and there are three fixed points.

space. Small oscillations about the stable fixed [redacted]

The final “fixed point” is the degenerate case $(p = P,\ \psi = \text{arbitrary})$ for which the wave am- [redacted]

bility results, as discussed in the previous sec- [redacted]

$$M_{\mathrm{e}} \equiv N_{\mathrm{m}} \frac{2J_0}{2J_0 - p_0}\,. \tag{41}$$

Here M_{e} is the effective mass due to the self- [redacted]

in a fixed potential well. This gives a bounce frequency of

$$\omega_{\mathrm{B}}^2 = 2\left(\frac{J_0}{N}\right)^{1/2}\left(1 - \frac{p_0}{2J_0}\right). \tag{42}$$

The first factor is just the bounce frequency of a particle in a fixed well, normalized in accord [redacted]

to zero, corresponding to the initial conditions $J(0) = p(0) = 0$. In this case $J(\tau) = -p(\tau)$ [redacted]

$$\Omega \equiv \frac{p_0}{N_{\mathrm{m}}} = -\left(\frac{N_{\mathrm{m}}}{N}\right), \quad \psi = 0\,. \tag{43}$$

From the simulations we will find $N_{\mathrm{m}} \simeq 0.4N$, and thus that

$$\Omega = -0.74, \quad |\Phi| = 0.29, \quad \omega_{\mathrm{B}} = 0.94\,. \tag{44}$$

3. Simulations

Simulations of the model of Eqs. (32) were [redacted]

plectic, leap frog method.

[redacted]

In the simulations the particles are initial- [redacted]

with $N = 10^4$. For small τ the wave ampli- [redacted]

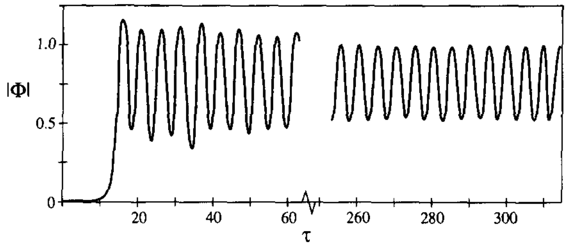

Fig. 2. Plot of $|\Phi(\tau)|$, the normalized wave amplitude, for $N = 10\,000$ particles initialized as a cold beam.

predicted by Eq. (36) with the phase velocity $v_\phi = \mathcal{R}(e^{2\pi i/3}) = -0.5$. As the wave grows, the beam experiences a growing sinusodal perturba-

[redacted]

begin to oscillate in the wave. Consequently the

[redacted]

the single-wave model does not allow the devel-

[redacted]

These would lead to the growth of other waves and undoubtedly greatly change the subsequent behavior of the system.

None-the-less, the subsequent development of the OWM dynamics is quite interesting. As the beam particles begin to oscillate in the wave, their oscillation frequencies depend upon their energy, just as for a single particle in a fixed potential. Thus as the beam begins to rotate about the potential minimum, those particles closer to the center have larger oscillation frequencies than those near the "separatrix".

If the wave amplitude were fixed, one would see phase mixing of the particles (visualized as an ever tighter spiral in the particle phase space), and the oscillations in the particle total

[redacted]

However since $v_\phi \approx -0.5$ and the beam is ini-

[redacted]

amplitude is not fixed and each beam particle

[redacted]

known, the phase space for a single beam par-

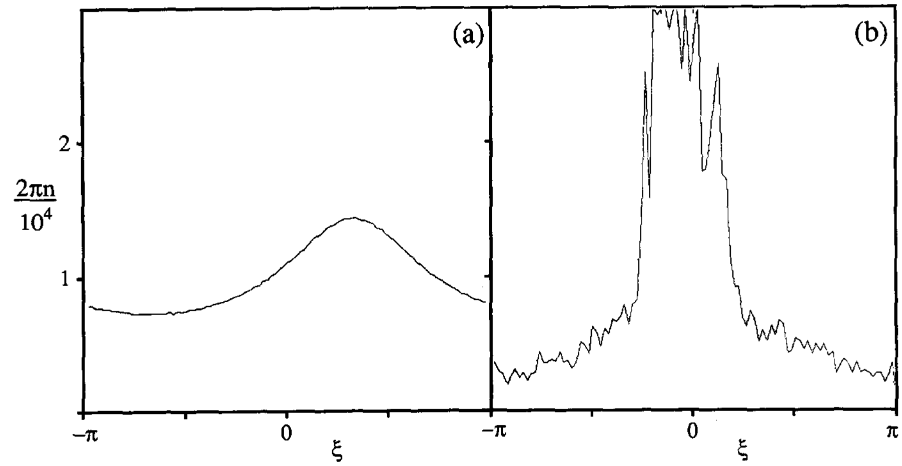

Fig. 3. Plot of the beam density as a function of position. The sinusoidal distortion of the density due to the growing wave

[redacted]

[redacted]

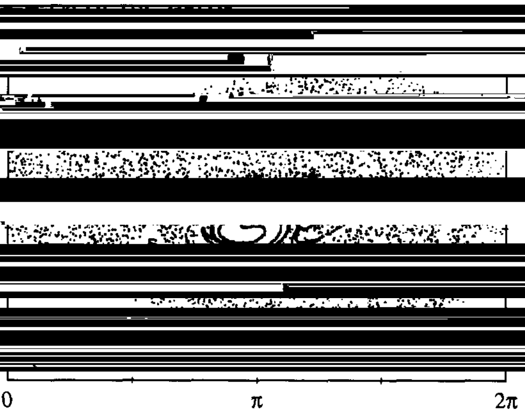

Fig. 4. Plot of the beam particle phase space at $\tau = 641$ showing a well defined macroparticle and chaotic sea.

[redacted]

ticle in a given oscillating potential has chaotic

[redacted]

zones, especially for the regions near the oscillat-

[redacted]

of the beam particles at a fixed time. Note the

[redacted]

with a nearly uniform density, and the other a more coherent cluster of particles. In the cluster one still sees evidence of the initial beam, though

[redacted]

In the simulations, which were carried out up

[redacted]

sisted, and indeed, as can be seen in Fig. 2 the oscillations become increasingly periodic with time. Furthermore as we varied N up to 10^5 and improved the integration accuracy, we noticed that these oscillations became more periodic and constant in amplitude as the number of particles increased and as the accuracy improved. Thus we believe that the asymptotic state is a periodic one.

Meanwhile, the particle phase space exhibits

[redacted]

particles—those with relatively large energies in the wave frame—experience chaotic motion, and spread out roughly uniformly in a region of phase space whose average width is $\Delta\omega = 4.7$. We called the chaotic particles the "stochastic

[redacted]

"chaotic sea". The remaining 40% of the parti-

[redacted]

wave, as seen in the sequence of phase space

[redacted]

instantaneous separatrix of the wave, that is

[redacted]

of the cluster oscillate 180° out of phase. This clump of particles is treated as a single particle, the macroparticle, in the model of Section 4.

In addition to the uniform cold beam, several different initial conditions have been partially

[redacted]

these cases some subset of the particles remained

[redacted]

system did not appear to settle into an equilibrium. Cold beams with nonzero momenta also lead to oscillations as was shown in [3], though

[redacted]

the momentum. We have not investigated this in-

[redacted]

initial conditions that will give rise to a periodic final state.

3.2. Test particle

To investigate further dynamics of the beam particles, consider the "test particle" motion of a single particle in a given oscillating potential. This is obtained from the nonself-consistent, one and one half degree-of-freedom Hamiltonian

[redacted]

$$H_t(p,\xi,\tau) = \tfrac{1}{2}p^2 - 2\left(\frac{J(\tau)}{N}\right)^{1/2}\cos(\xi-\theta(\tau)), \tag{45}$$

where J and θ are considered to be given peri-

[redacted]

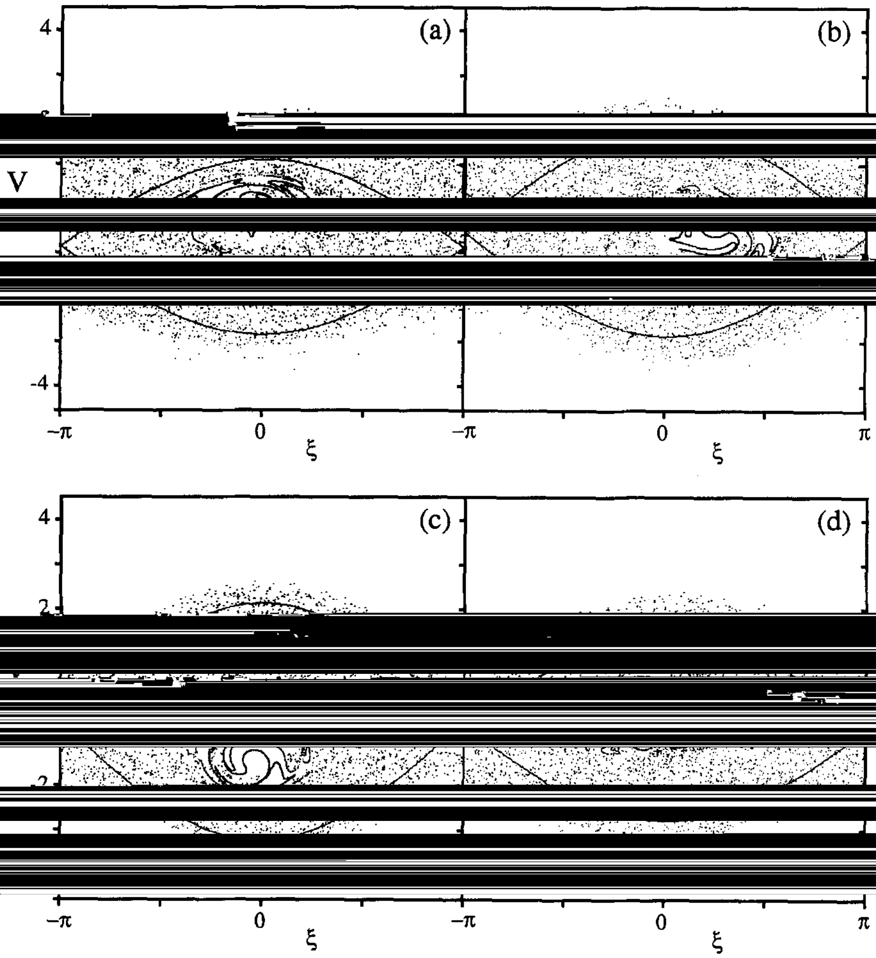

Fig. 5. Sequence of beam phase space plots over one bounce period. Note that the macroparticle bounces coherently in the [illegible]

[illegible]

Here we determine J and θ numerically, from the simulations of Section 3.1, building these functions from an average over a number of pe- [illegible]

A stroboscopic plot of the test particle dynamics is shown in Fig. 6 for several different values of θ. The dots represent the trajectories of a number of different test particles. As was also noted in [6], there is a prominent stable island in the test particle phase space which oscillates exactly out of phase with the potential; much of the rest of the phase space is chaotic. Also shown [illegible] in the plots represents the position of one of the 10 000 beam particles. Note that the macroparticle clump sits, as near as can be ascertained, [illegible] verifies an assertion in [6], where it was merely noted that some fraction of the beam particles initially stretched across the position of the test particle island.

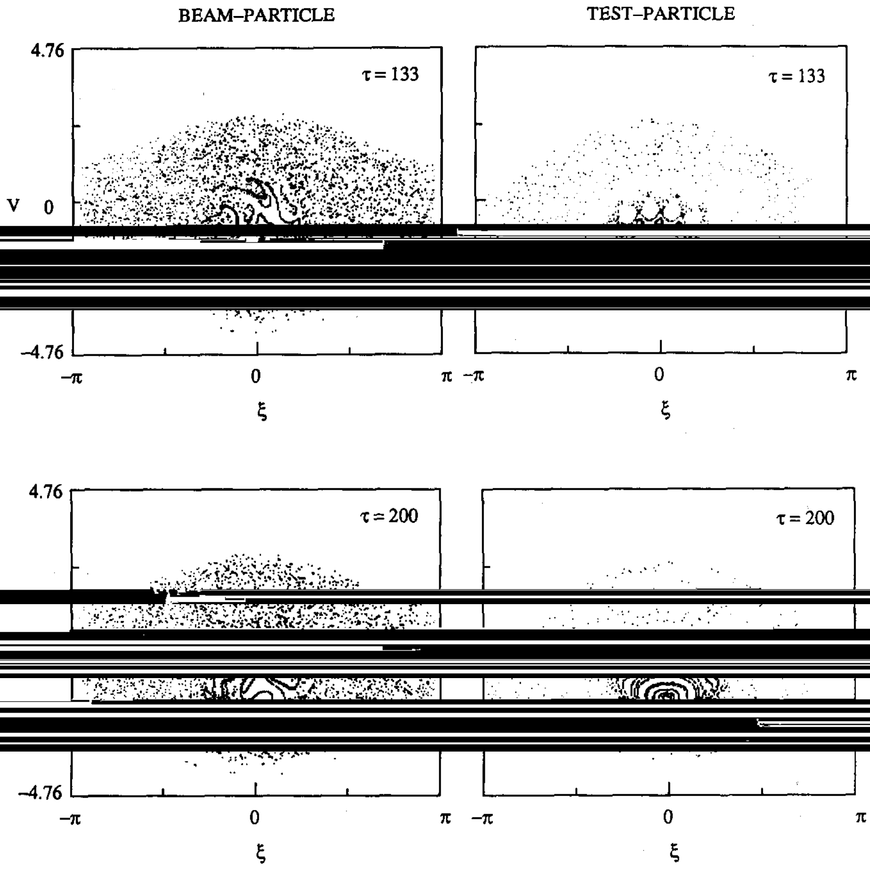

Fig. 6. Phase space plots comparing the full simulation of Section 3 with the dynamics of a test particle in a given time-dependent potential $\Phi(\tau)$ as determined by the simulation. Shown are several test particle initial conditions at three different times during a cycle.

4. Chaotic sea model

As we have seen from the simulations, the asymptotic state of the cold beam initial condition, evolved under the OWM Hamiltonian, appears to be almost exactly periodic. Approximately 40% of the initial beam forms a clump of particles that oscillates in the potential well formed by the wave. The remaining particles [illegible]

In this section we use these ideas to obtain a reduced Hamiltonian model of four degrees of freedom, which approximately describes the full 10 001 degree-of-freedom system. In the model, as noted above, we assume that the clump of regularly oscillating particles is localized enough so that all these particles can be treated as one located at (ξ, p). This macroparticle contains N_m particles and hence a mass mN_m and charge $-eN_m$. The approximation that the N_m particles can be treated as a single particle ignores any in- [illegible] be similar to the analysis of Mynick and Kaufman [5] who assumed that the cluster of particles formed a "bar" in phase space (an assump-

[redacted]

can be treated as a continuum with a constant

[redacted]

$$n_c(x,t) = \int_{v_-}^{v_+} f_c\,\mathrm{d}v = f_c(v_+ - v_-) \quad (46)$$

[redacted]

ber of such particles in the length L will be denoted by $N_c = N - N_m$. Particles in the chaotic sea evolve according to Eq. (2), and hence f_c evolves according to the Vlasov equation. As is

[redacted]

two equations for the evolution of the boundaries [18]. These equations are called the waterbag equations:

$$\frac{\partial v_+}{\partial t} + v_+\frac{\partial v_+}{\partial x} = -eE,$$
$$\frac{\partial v_-}{\partial t} + v_-\frac{\partial v_-}{\partial x} = -eE. \quad (47)$$

Following the philosophy of the derivation of

[redacted]

in $v_\pm(x)$:

$$v_\pm = v_\pm^0 + \tilde{v}_\pm\,\mathrm{e}^{ikx} + \tilde{v}_\pm^*\,\mathrm{e}^{-ikx}. \quad (48)$$

The equations of motion then become

$$\left(\frac{\partial}{\partial t} + \mathrm{i}kv_\pm^0\right)\tilde{v}_\pm = -eE_k, \quad (49)$$

where $v_\pm^0$ are the mean velocities (which do not

[redacted]

$$\frac{}{k\epsilon'}\left(f_c(\tilde{v}_+ - \tilde{v}_-) + \frac{N_m}{L}\,\mathrm{e}^{-ikx_m}\right). \quad (50)$$

[redacted]

$$\omega_\pm \equiv \frac{kv_\pm^0 - \omega_0}{\omega_b}, \quad V_\pm \equiv \frac{k}{\omega_b}\tilde{v}_\pm\,\mathrm{e}^{\mathrm{i}\omega_0 t}. \quad (51)$$

In terms of these variables the equations of mo-

[redacted]

$$\dot{\Phi} = \mathrm{i}\frac{N_c}{N\Delta\omega}(V_+ - V_-) + \mathrm{i}\frac{N_m}{N}\,\mathrm{e}^{-\mathrm{i}\xi},$$
$$\ddot{\xi} = \mathrm{i}\Phi\,\mathrm{e}^{\mathrm{i}\xi} - \mathrm{i}\Phi^*\,\mathrm{e}^{-\mathrm{i}\xi},$$

[redacted]

where $\Delta\omega = \omega_+ - \omega_-$ is the average, nondimensional width of the chaotic sea.

This set of equations is also a Hamiltonian system, with the wave action–amplitude variables defined in Eq. (28), and the new action–amplitude variables for the chaotic sea defined by

$$V_+ = \left(\frac{J_+\Delta\omega}{N_c}\right)^{1/2}\mathrm{e}^{-\mathrm{i}\theta_+},$$

[redacted]

which results in the Poisson bracket relations

$$[V_\pm^*, V_\pm] = \pm\mathrm{i}\frac{\Delta\omega}{N_c}. \quad (54)$$

The Hamiltonian takes the form

$$H = \frac{p^2}{2N_m} + \frac{N_c\omega_+}{\Delta\omega}|V_+|^2 - \frac{N_c\omega_-}{\Delta\omega}|V_-|^2$$

[redacted]

$$+ N_m\Phi^* \, e^{-i\xi} + \text{c.c.}\Big\} .$$

[redacted]

In terms of canonical variables this becomes

[redacted]

$$H = \frac{p^2}{2N_m} + \omega_+ J_+ - \omega_- J_-$$

[redacted]

where the coefficients are given by

[redacted]

$$\alpha = \left(\frac{N_c}{N\Delta\omega}\right)^{1/2} , \quad \beta = \frac{N_m}{\sqrt{N}} . \tag{57}$$

The first three terms in the Hamiltonian repre-

[redacted]

of macroparticle energy and harmonic terms for the oscillations of the chaotic sea boundary. The last three terms give the electrostatic interaction energy.

Thus we have reduced the 10001 degree-of-freedom Hamiltonian to one of four degrees of freedom, which describes well the motion in the periodic final state of the simulations, *provided* the three parameters ω_+, ω_- and N_m are given.

[redacted]

tal momentum given by,

[redacted]

tum in the wave and macroparticle, and the last three are the contributions of the chaotic sea. These latter terms include the momentum in the oscillations of the waterbag boundary, $J_+ - J_-$,

[redacted]

tion can also be derived from Eq. (33) by split-

[redacted]

served quantities besides the energy. Thus the

[redacted]

and should exhibit the full complexity of such

[redacted]

diffusion.

The stable equilibrium of Eq. (56) corre-

[redacted]

$$J_\pm = \frac{\alpha^2}{(\Omega - \omega_\pm)^2} J . \tag{59}$$

[redacted]

$N_m \approx 0.4N$, $\omega_+ \approx 1.9$, and $\omega_- \approx -2.8$. This gives $\alpha = 0.36$, $\beta = 40$, and $\Delta\omega = 4.7$. For these parameters we can solve Eq. (59) to obtain

$$\Omega = -0.66 , \quad |\Phi| = 0.73 ,$$

$$|V_+| = 0.29 , \quad |V_-| = 0.35 . \tag{60}$$

On the other hand, using Fig. 5 to determine the average phase velocity of the wave in the simulations we obtain $\Omega \approx -0.67$. The average value of $|\Phi|$ from Fig. 2 is 0.75. Both of these

[redacted]

Ω as given, then Eq. (59) gives the macroparticle

[redacted]

To compute the frequency of small oscillations about the equilibrium, we first eliminate the action J using the conservation law (58), defining phases $\psi = \theta - \xi$, and $\psi_\pm = \theta_\pm \mp \theta$, conjugate

[redacted]

$$\delta^2 H = \tfrac{1}{2}\delta p \cdot \mathbf{M}^{-1} \cdot \delta p + \tfrac{1}{2}\delta\psi \cdot \mathbf{K} \cdot \delta\psi \qquad (61)$$

[redacted]

$\delta\psi_+, \delta\psi_-$) are the deviations from equilibrium.

[redacted]

itive definite. The matrix **K**, the effective spring constant matrix, turns out to be diagonal. In order that the three terms in **K** be positive, we must assume that $\psi = \psi_+ = 0$, while $\psi_- = \pi$. This is consistent with the fact that the lower boundary of the chaotic sea is observed to have a 180° phase lag with respect to the upper boundary.

The frequencies of small oscillation are given by the square roots of eigenvalues of the matrix $\mathbf{KM}^{-1}$. For the parameters of the simulation, the mass matrix is diagonal to a good approximation. The element M_{11}^{-1} turns out to be identical to $1/M_e$ of Eq. (41); neglecting terms of order $J_\pm/J$, the other diagonal elements are

[redacted]

The matrix **K** is

[redacted]

Using the values obtained before for the equilibrium, we determined the eigenvalues numerically from the full matrix. The three oscillation frequencies are

[redacted]

modes; however, it might be possible to determine these through careful simulation.

[redacted]

much better than the single particle calculation

[redacted]

electrostatic interaction of many particles with a plasma wave. The wave arises from an instability (the beam–plasma instability) of the initial state corresponding to a cold beam of particles. In the simulations, the asymptotic state corresponds to a periodically oscillating wave amplitude together with a trapped clump of particles. About 42% of the particles are trapped by an approximate invariant surface within the oscillating wave, while the remaining particles move chaotically—becoming successively trapped and detrapped.

We modelled this motion by a four degree-of-

[redacted]

chaotic sea, and one to the wave. This model quantitatively captures the asymptotic state of

[redacted]

One would like to speculate that there are other physical systems for which the effect of self-consistency would be similar. For example in the case of galactic dynamics, the self-

[redacted]

discussed at the end of Section 3.1.

– Is there a way of self-consistently calculating

[redacted]

[redacted]

Is there a periodic state of the many particle sys-

[redacted]

the effect of adding additional harmonics?

[redacted]

the full array of possible Hamiltonian motions.

[redacted]

manifolds leading to homoclinic phenomena.

[redacted]

states that correspond to these motions?

Acknowledgements

[redacted]

was provided by the US Dept. of Energy Contract No. DE-FG05-80ET-53088 to the Univer-

[redacted]

References

[redacted]

plasma, Phys. Fluids 14 (1971) 1204–1212.

[redacted]

monoenergetic electron beam, Plasma Phys. 14 (1972) 591–600.

[redacted]

motion in a large amplitude plasma wave, Phys. Fluids

[redacted]

nonlinear Langmuir waves, Phys. Fluids 24 (1981) 260–267.

[redacted]

ear diffusion far from the chaotic threshold, Physical

[redacted]

three-wave interaction: integrable case of this system

[redacted]

[10] J.R. Cary, I. Doxas, D.F. Escande and A.D. Verga, Enhancement of the velocity diffusion in longitudinal plasma turbulence, Phys. Fluids B 4 (1992) 2062.

[11] P.J. Morrison, The Maxwell–Vlasov equations as a continuous Hamiltonian system, Phys. Lett. A80 (1980) 383–386.

[redacted]

Princeton Plasma Physics Laboratory Report PPPL-

[redacted]

Publication Service, 335 East 45th Street, New York, NY 10017).

[13] A.N. Kaufman and R.L. Dewar, Canonical derivation of the Vlasov-Coulomb noncanonical Poisson bracket,

[redacted]

plasma energy, and Hamiltonian action–angle variables for the Vlasov equation, Phys. Fluids B4 (1992) 3038–3057.

[redacted]

[17] T.M. O'Neil, Collisionless damping of nonlinear plasma

[redacted]

(Academic, New York, 1972), pp. 45-47.