



3. Let  $N > 0$  be an unknown integer and let  $X_1, \dots, X_n$  be a random sample from the distribution with

$$P(X_i = k) = \begin{cases} \frac{1}{2N+1} & ; k = 0; 1; 2; \dots; N; \\ 0 & ; \text{otherwise:} \end{cases}$$

- (a) Show that  $M = \max\{X_1, \dots, X_n\}$  is a sufficient statistic for  $N$ .
- (b) Show that  $M$  is a complete statistic for  $N$ .
- (c) Determine the uniformly minimum variance unbiased estimator (UMVUE) for  $N$ . Simplify your answer!
- Hint for part (c). Determine constants  $a, b, c$  such that  $aM + b \mathbb{I}[X_1 = 0] + c$  is unbiased for  $N$ , where  $\mathbb{I}[X_1 = 0]$  is notation for the indicator function of the event  $[X_1 = 0]$ .
4. Suppose that  $X_1$  is a random sample of size 1 from the distribution with probability density function

$$f(x; \theta) = \begin{cases} \frac{1}{2} & ; \text{if } 0 < x < \theta; \\ \frac{1}{2x^2} & ; \text{if } x > \theta; \\ 0 & ; \text{otherwise:} \end{cases}$$

- For which values of  $0 < \theta < 1$  does  $H_0 : \theta = 1$  versus  $H_1 : \theta < 1$  admit a unique uniformly most powerful (UMP) test of size  $\alpha$ ? Specify the rejection region associated with each of those tests.
5. Let  $0 < \lambda < \mu$  be real constants and consider an M/M/1 queueing system with arrival rate  $\lambda$  and service rate  $\mu$ , which is initially empty. Suppose that the server is turned ON and OFF according to the following rules:
- it remains OFF as long as the number of customers in the system is less than 2,
  - it is turned ON as soon as the number of customers in the system becomes 2 and then remains ON until completely emptying the system of customers.
- Based on the above, please respond:
- (a) Model the number of customer in the system as a continuous time Markov chain with state space  $\{0, 1', 1, 2, 3, \dots\}$ , where  $1'$  represents the configuration of having exactly one customer in the system while the server is OFF, and state 1 represents the same but while the server is ON. Represent the rate transition matrix of the chain as a directed graph with weighted edges.
- (b) Show that this system has a stationary distribution and determine it explicitly.
- (c) After a long time of operation, what is the probability that a new customer encounters the queue empty?