Onset of synchronization in large networks of coupled oscillators

Juan G. Restrepo,[1,2,*] Edward Ott,[1,3] and Brian R. Hunt[2,4]
[1]*Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, Maryland 20742, USA*
[2]*Department of Mathematics, University of Maryland, College Park, Maryland 20742, USA*
[3]*Department of Physics and Department of Electrical and Computer Engineering, University of Maryland, College Park, Maryland 20742, USA*
[4]*Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742, USA*
(Received 8 November 2004; published 30 March 2005)

We study the transition from incoherence to coherence in large networks of coupled phase oscillators. We present various approximations that describe the behavior of an appropriately defined order parameter past the transition and generalize recent results for the critical coupling strength. We find that, under appropriate conditions, the coupling strength at which the transition occurs is determined by the largest eigenvalue of the adjacency matrix. We show how, with an additional assumption, a mean-field approximation recently proposed is recovered from our results. We test our theory with numerical simulations and find that it describes the transition when our assumptions are satisfied. We find that our theory describes the transition well in situations in which the mean-field approximation fails. We study the finite-size effects caused by nodes with small degree and find that they cause the critical coupling strength to increase.

DOI: 10.1103/PhysRevE.71.036151 PACS number

II. SELF-CONSISTENT ANALYSIS

As shown by Kuramoto [6], the dynamics of weakly coupled, nearly identical limit cycle oscillators can, under certain conditions, be approximated by an equation for the phases θ_n of the form

$$\dot{\theta}_n = \omega_n + \sum_{m=1}^{N} \Gamma_{nm}(\theta_m - \theta_n), \qquad (2)$$

where ω_n is the natural frequency of the oscillator n, N is the total number of oscillators, and Γ_{nm} is a periodic function depending on the original equations of motion. The all-to-all Kuramoto model assumes that $\Gamma_{nm}(\theta_m - \theta_n) = (k/N)\sin(\theta_m - \theta_n)$, where k represents an overall coupling strength. In order to incorporate the presence of a heterogeneous network, we assume that $\Gamma_{nm}(\theta_m - \theta_n) = kA_{nm}\sin(\theta_m - \theta_n)$, where $A_{nm} \geq 0$ are the elements of an $N \times N$ adjacency matrix A determining the connectivity of the network. Therefore, we study the system

$$\dot{\theta}_n = \omega_n + k\sum_{m=1}^{N} A_{nm}\sin(\theta_m - \theta_n). \qquad (3)$$

For specificity, we will primarily consider the case where the A_{nm} are either 0 (nodes n and m are not connected) or 1 (nodes n and m are connected, and all connections have equal strength). We assume that the network is undirected, so that $A_{nm} = A_{mn}$. We assume also that, for each n, the corresponding ω_n is independently chosen from a known oscillation frequency probability distribution $g(\omega)$. We assume that $g(\omega)$ is symmetric about a single local maximum (cf. Sec. V), which without loss of generality we can take to be at $\omega = 0$. (If the mean frequency is $\omega_0 \neq 0$, we make the change of coordinates that shifts each ω_n by ω_0 and each θ_n by $\omega_0 t$.) In this case, synchronization will occur at frequency 0; i.e., θ_n will remain approximately constant for synchronized nodes.

We define a positive real-valued local order parameter r_n by

$$r_n e^{i\psi_n} \equiv \sum_{m=1}^{N} A_{nm}\langle e^{i\theta_m}\rangle_t, \qquad (4)$$

where $\langle \cdots \rangle_t$ denotes a time average. In terms of r_n, Eq. (3) can be rewritten as

$$\dot{\theta}_n = \omega_n - kr_n\sin(\theta_n - \psi_n) - kh_n(t),$$

the real part equation 8) can be neglected because of the symmetry of g about 0. We thus obtain the approximation

$$r_n = \sum_m \frac{1}{kr_m} A_{nm} \cos(\,_m - \,_n) \sqrt{1 - \left(\frac{\,_m}{kr_m}\right)^2}. \qquad (10)$$

Since we are interested in the transition to coherence, we look for the solution of Eq.

$$r_n = r_n^{0} + \quad r_n, \qquad (19)$$

where

expansion is appropriate for 5. For 3 5, he obtains in the limit $N\to$ that r scales near the transition as r $k/k_{mf}-1|^{1/ \; -3|}$. A similar situation occurs in the perturbation theory Eqs. 22| and 23|[, which was also based on expanding g to second order. According to the previous discussion, we will only use the expression for r obtained from the perturbation theory for situations in which d^4 is finite.

large N (too large for us to simulate). In fact, as $N\to\infty$, diverges while $\langle d^2\rangle/\langle d\rangle$ remains finite. Thus, the critical coupling constant obtained from our theory approaches zero as $N\to\infty$, while the one obtained from the mean-field theory remains constant. This suggests that the few nodes with high degree are able, for large enough N, to synchronize the network and that these nodes are not taken into account by the mean-field theory.

For $\gamma > 3$, we observe from Fig. 2 that is less than $\langle d^2\rangle/\langle d\rangle$ when $N=5000$. Thus, in this range, the mean-field theory predicts a transition for a coupling constant that is *smaller* than that predicted by the perturbative approach. In the next section we will show, for a numerical example in this regime, that the transition occurs for a larger coupling than that predicted by the mean-field theory.

III. EXAMPLES

In order to test the results in Sec. II, we choose a distribution for the natural frequencies given by $g(\omega)=3/4(1-\omega^2)$ for $-1<\omega<1$ and $g(\omega)=0$ otherwise. In order to generate the network, we specify a degree distribution and we use the "configuration" model (e.g., Sec. 4.2.1 of Ref. [1] and references therein) to generate a random network realization with the specified degree distribution: (i) we first generate a *degree sequence* by assigning a degree d_n to each node n according to the given distribution; (ii) imagining that each node n is given d_n spokes sticking out of it, we choose pairs of spoke ends at random and connect them.

We consider a fixed number of nodes, $N=2000$, and the following networks with uniform coupling strength (i.e., $A_{nm}=1$ or 0): (i) the degrees are uniformly distributed between 50 and 149, and (ii) the probability of having a degree d is given by $p(d) \propto d^{-\gamma}$ if $50 \le d \le 2000$ and $p(d)=0$ otherwise, where γ is taken to be 2, 2.5, 3, and 4. Our choice $p(d)=0$ for $d<50$ ensures that there are no nodes of small degree and suggests that our approximation of neglecting the noiselike, fluctuating quantity h_n in Eq. (5) is valid. We return to this issue in Sec. VI.

The initial conditions for Eq. (3) are chosen randomly in the interval $[0,2\pi]$ and Eq. (3) is integrated forward in time until a stationary state is reached (stationary state here means stationary in a statistical sense; i.e., the solution might be time dependent but its statistical properties remain constant in time). From the values of $\theta(t)$ obtained for a given k, the order parameter r is estimated as $r \approx \langle |\sum_{m=1}^{N} d_m e^{i\theta_m}|\rangle_t / \sum_{m=1}^{N} d_m$, where the time average is taken after the system reaches the stationary state. (Close to the transition, the time needed to reach the stationary state is very long, so that it is difficult to estimate the real value of r. This problem also exists in the classical Kuramoto all-to-all model.) The value of k is then increased and the system is allowed to relax to a stationary state, and the process is repeated for increasing values of k.

In Fig. 3 we show the results for the network with a uniform degree distribution as described above [network (i)]. We plot r^2 from numerical solution the full system in Eq. (3) (triangles), the theoretical prediction from the time-averaged theory (solid line), and the prediction from the mean-field theory (long-dashed line) and from the perturbation theory (short-dashed line) (see Table I) as a function of k/k_c, where k_c is given by Eq. (16). The frequency distribution approximation agrees with the time-averaged theory, so we do not include it in the plot. In this case, all the theoretical predictions provide good approximations to the observed numerical results. The time-averaged theory reproduces remarkably well the numerical observations. Even the irregular behavior near the transition is taken into account by the time-averaged theory. The mean-field theory is in this case a good approximation, providing a fair description of the order parameter past the transition. The perturbation theory is valid in this case up to $k/k_c \approx 1.3$.

The results for the networks with power-law degree distributions [networks (ii)] are shown in Figs. 4(a), 4(b), 4(c), and 4(d) for $\gamma=2$, 2.5, 3, and 4, respectively. The order parameter r^2 from numerical solution of the full system in Eq. (3) (triangles), the time-averaged theory (solid line), the frequency distribution approximation (stars), and the mean-field theory (long-dashed line) sharper transition than actually occurs. The mean-field approximation agrees closely with the frequency distribution approximation for $=4$ and, away from the transition, for $=3$. However, for $=2$ and $=2.5$, it deviates greatly from the other approximations and from the numerical simulation. The critical coupling strengths predicted by the mean-field theory and by the perturbation theory are very close for $=4$, but the mean-field theory predicts a transition at about 10% smaller coupling for $=3$, about 20% smaller for $=2.5$, and about 40% smaller for $=2$. Since the transition in the numerical simulation is not so well defined, both approximations are reasonable for $=3$, but for $=2$ and $=2.5$ the critical coupling strength predicted by the mean-field approximation is clearly too small.

In the past years, it has been discovered that many real-world networks have degree distributions which are power laws with exponents between 2 and 3.5 [1,2,15]. In order to accurately predict the critical coupling strength across this range of exponents, the critical coupling constant given by $k_c = k_0/$ determined by the largest eigenvalue of the adjacency matrix should be used. The behavior of the order parameter can be estimated using the time-averaged theory or the frequency distribution approximation. These two approximations were found to be consistently accurate for the range of exponents and values of the coupling constant studied. For the value of N used, the mean-field theory works well in predicting the critical coupling strength and the behavior of the order parameter if one is interested in values of larger than 3.

Tables II and III present the results of comparing the theoretical predictions with the numerical integration of Eq. (3) for different networks. Table II compares the observed critical coupling strength with the theoretical estimate. If both are close, the entry is "G," and otherwise "NG." Table III compares the predicted behavior of the order parameter past the transition with the observed one. If the corresponding entry in Table II is "NG," no comparison is attempted. The entries are the range of k/k_c over which the corresponding

condition, which we assume to be randomly drawn from $0,2$ |. In this section, by $\langle \cdots \rangle$ we mean an expected value—i.e., an ensemble average, rather than an average over t or n.

valid in this case, we find that it correctly describes the trend present in the numerical observations—i.e., a shifting of the transition to coherence to larger values of the critical coupling as nodes of small degree become important.

VII. DISCUSSION

A transition to coherence in large networks of coupled oscillators should be expected at a critical value of the coupling strength which is determined by the largest eigenvalue of the adjacency matrix of the network and its associated eigenvector. In the all-to-all case, the largest eigenvalue is $N-1 \approx N$ and thus the Kuramoto result $k_c = k_0/N$ is recov-

$$\sum_{m \in kr_m}^{N} A_{nm'} e^{i\omega_m t} . \qquad (A1)$$

We will follow to some extent Chap. 12 of Ref. 4[. The time average is given by

$$\langle e^{i\omega_m t} \rangle = \int_{-}^{} e^{i\omega} p_m(\omega|\ldots) d\omega , \qquad (A2)$$

where $p_m(\omega|\ldots) d\omega$ is, given the connections of node m and its natural frequency ω_mm02 Tf 9.978 0 1.882 9.978 214.255j 7460 40 165v

$$b_j = \frac{k}{2}\sum_{n=1}^{N}\frac{A_{jn}b_n}{s - i\omega_n} + \frac{k}{2}e^{2\,\mathrm{Im}(s)t}\sum_{n=1}^{N}\frac{A_{jn}b_n^* e^{2i\omega_n^0 t}}{s^* + i\omega_n}. \qquad (C6)$$

The second sum is very small due to the incoherence of the ω_n^0's. So, changing indices, we are left with the eigenvalue equation

$$b_n = \frac{k}{2}\sum_{m=1}^{N}\frac{A_{nm}b_m}{s - i\omega_m}, \qquad (C7)$$

as claimed in Sec. V.

If, as proposed in Sec. VI, there are fluctuations in the values of $\omega_n^0(t)$ such that $\omega_n^0(t) = \omega_n t + \phi_n + W_n(t)$, where $W_n(t)$ is a random walk such that $\langle W_n(t)\rangle = 0$ and $\langle W_n(t)^2\rangle = 2D_n t$, we take the expected value of Eq. (C5). We use the fact that for a Gaussian random variable x with variance σ^2 we have $\langle e^{ix}\rangle = e^{i\langle x\rangle - \sigma^2/2}$. In this case, $x = \omega_n(t' - t)$ and $\sigma^2 = 2D_n|t - t'|$. We obtain, after performing the integration,

$$b_n = \frac{k}{2}\sum_{m=1}^{N}\frac{A_{mn}b_m}{s + D_m - i\omega_m}. \qquad (C8)$$

[1] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[2] A.-L. Barabási and R. Albert, Rev. Mod. Phys. **74**, 47 (2002).
[3] A.-L. Barabási and R. Albert, Science **286**, [illegible]