A draft of INRIA lectures, May 1991

Wavelets, Multiresolution Analysis and Fast Numerical Algorithms

Bey n

M po e o eq e o de N ope on o co p e e

$$p_j = \sum_{i \, j} \frac{q_i q_j}{| \, i}$$

e e e od y e e ed de ce fo ed c n p d en eq on o
p e ne y e fo e co of n n e en y cond on n e of e e n
ce f n e d of n e d ence o n e e e en ep e en on e e e ep
e en on of e de e n e e e en p e od c on

e n y e

II.1 Multiresolution analysis.

e e de n on of e e o on n y no on n od ced y Meye nd M c p e e e en f ee n en e of e en o o of ene y e o e e ne e n e d of e e en e

$$\mathbf{V_n} \subset \quad \subset \mathbf{V} \subset \mathbf{V} \subset \mathbf{V} \quad \mathbf{V} \subset \mathbf{L} \; \mathbf{R^d}$$

n n e c e z on e p ce **V** n e d en on

II.2 The Haar basis

o on e e on y e p e of e e o on n y y n De n on Cond on e Co p n n o p e of e o e de e n e ec e nd p o de f p o o ype fo n e c e pe en on f d en $_{j;k}$ $j=$ j $\in$ **Z** fo ed y e d on nd n on of n e f nc on

$$\begin{cases} & \text{fo} \\ - & \text{fo} \quad \le \\ & \text{e e e} \end{cases} \qquad 7$$

n c e e e e c c e c f nc on of e n e o e c $_{j;k}$ $j=$ j $-$ $\in$ **Z** e of $\mathbf{V_j}$ nd $_{j;k}$ $j=$ j $-$ $\in$ **Z** e of $\mathbf{W_j}$

e deco po on of f nc on n o n o de N p oced e en N n p e of f nc on c y fo p c y e o of e of ed e e of f on n e of en n

$$_{k} = {}_{n=} \int_{-n k}^{-n k+)} f \; d$$

e o n coe c en

$$d_k^{j+} = \frac{}{\sqrt{}} \; {}_k^j - {}_k^j$$

nd e e

$$_k^{j+} = \frac{}{\sqrt{}} \; {}_k^j \quad {}_k^j$$

fo $n-$ nd n j $-$ e y o e e n e o e of coe c en d_k^j $_k^j$ n eq e $N-$ dd on nd N p c on n o d en on e e e o n o e nd d fo e econd de ned y e e of ee nd of f nc on ppo ed on e $_{j;k}$ $_{j;k'}$ y $_{j;k}$ $_{j;k'}$ y nd $_{j;k}$ $_{j;k'}$ y e e e c c e c f nc on of e n e nd $_{j;k}$ = $_{j=}$ $_{j}$ — ep e en n n ope o n e d o e non nd d fo e e no o y eco e c e e

By con de n n n e ope o

$$f \quad = \int \quad y \; f \; y \; dy$$

nd e p nd n e ne n o d en on e nd fo C de on Zy nd nd p e do d en ope o e dec y of en e f nc on of e d nce f o e d on f e n e e ep e en on n n e o n e ne e e c of ope o e en y n e o d on e ne e oo y f o e d on o e p e e ne y of C de on Zy nd ope o fy e e e

$$| \quad y| \leq \frac{}{| \; - y|}$$

$$|^{M}_{x} \quad y| \quad |^{M}_{y} \quad y| \leq \frac{C_{M}}{| \; - y|^{\;+M}}$$

fo e $M \geq$ Le M = n nd con de

$$^{j}_{kk'} = \int\int \quad y \quad _{j;k} \quad _{j;k'} \; y \; d \; dy$$

e e e e e d nce e een $| \; - \; '| \geq$ nce

$$\int \quad _{j;k} \quad d \; = \qquad\qquad e$$

e e

$$| \qquad y \qquad f \; r \quad x$$

e on e n n e dec y n cen o e co p n n p c c o e f e dec y nece y o e f nc on e e n n o en e n n o en e e pon e fo n n p c c o e con o n e con n n e co p e y e e of e f o

II.3 Orthonormal bases of compactly supported wavelets

e q e on of e e ence of e on n y oo f nc on of Cond on e no e ed n e con c on of e o ono e e ene z n f nc on y o e . , nd Meye . , e con de econd e o of on y of { − }k Z p e

$$\mathbf{k} \quad - \int_{\mathbb{Z}}^{+} \quad - \quad d \quad - \int_{\mathbb{Z}}^{+} | \quad | \; e^{i\mathbf{k}} \; d$$

nd e efo e

$$\mathbf{k} \quad - \int_{\mathbb{Z}} \sum_{\mathbf{l} \in \mathbb{Z}} | \quad \quad | \; e^{i\mathbf{k}} \; d$$

nd

$$\sum_{\mathbf{l} \in \mathbb{Z}} | \quad \quad | \quad -$$

n e o n

$$\sum_{\mathbf{l} \in \mathbb{Z}}$$

no d c o o e e . , , . , . . , on e c ed e ∈ **Z** e e e { j;k — j= j — }k Z fo n o ono of **W**j

e fo o n e D ec e , c c e ze ono e c po yno on of c co e pond o e o ono e of co p c y ppo ed e e n n o en

Lemma II.1 Any trigonometric polynomial solution of (2.26) is of the form

h — ½ ei i M ei

where $M \geq$ **is the number of vanishing moments, and where is a polynomial, such that**

| ei | —P n ½ n M ½ ½ co

where

$P\ y$ — k M k $M -$! y^k e

and is an odd polynomial, such that

$\leq P\ y \quad y^M$ ½ — d f

e e $^{j}_{k}$ nd d^{j}_{k} y e e ed pe od c eq ence e pe od n j Co p n

nd ed y e py d e e e

$$
\begin{array}{ccccccc}
\{\ {}_{\mathbf{k}}\} & \longrightarrow & \{\ {}_{\mathbf{k}}\} & \longrightarrow & \{\ {}_{\mathbf{k}}\} & \longrightarrow & \{\ {}_{\mathbf{k}}\} \ \cdots \\
 & \searrow & & \searrow & & \searrow & \\
 & & \{d_{\mathbf{k}}\} & & \{d & &
\end{array}
$$

e en de ne $f_{\mathbf{m}} := f_{\mathbf{m}} - \mathbf{m} \cdot f$ e e $\mathbf{m}$ c o en o $\langle f_{\mathbf{m}} \; \mathbf{M} \rangle :=$ fo
$:=$ M c e n e de ed o o on y o $\mathbf{M}$ y e con n e o

e e $\mathbf{V}_j^{M}$ en n e nd e p ce $\mathbf{W}_j^{M;}$ e o on co p e en of $\mathbf{V}_j^{M;}$ n $\mathbf{V}_j^{M;}$

$$\mathbf{V}_j^{M;} = \mathbf{V}_j^{M;} \; M \; \mathbf{W}_j^{M;}$$

e p ce $\mathbf{W}^{M;}$ p nned y e o ono

$$\{ \; i \quad l\, y \quad i \quad l\, y \quad i \quad l\, y \quad = \quad M \}$$

e e $\{ \; {}_m\}_m^{m \; M}$ n o ono fo $\mathbf{V}^{M}$ c e e en on e M e e en n n o en e o on o e po yno $^i y^l$ = $M -$

e p ce $\mathbf{W}_j^{M;}$ p nned y d on nd n on of e f nc on of $\mathbf{W}^{M;}$ nd e of $\mathbf{L}$ con of e f nc on nd e o o de po yno $^i y^l$ = $M -$

e no e e o d en on e e eq e M d en co n on of one d en on f nc on e e M e n e of n n o en On e o e nd e o d en on o ned y n co p c y ppo ed e e eq e on y ee c co n on c p e e con c on of e non nd d fo e ec on

II.5 A remark on computing in the wavelet bases

n y e no e once e e een c o en co p e e y de e ne e f nc on nd nd e efo e e e on n y n n e e n o e on n p ope y con c ed o e f nc on nd e ne e co p ed D e o e ec e de n on of e e e e n p on e pe fo ed e q d e o e nd e en f ey n o e q n e c ed nd A n e p e e co p e e o en of e n f nc on e e p e on fo e o en

$$\mathcal{M}^m = \int {}^m \; d \quad = \quad M -$$

n e of e e coe c en $\{ \; {}_k\}_k^{k \; L}$ y e fo nd n fo fo

$$= \prod_j \; {}^j$$

e e

$$= \sum_k {}^k \; {}_k e^{ik}$$

e o en $\mathcal{M}^{m}$ e o ned n e c y n e de ed cc cy y ec e y ene n eq ence of ec o $\{\mathcal{M}^{m}_{r}\}^{M}_{m}$ fo r =

$$\mathcal{M}^{m}_{r+} = \sum_{j}^{m} \quad ! \quad j\, r\, \mathcal{M}^{m}_{r} \; j\, \mathcal{M}^{j}$$

n

$$\mathcal{M}^{m} = m^{\frac{1}{2}} \sum_{k}^{k} k^{m} = \quad M -$$

c ec o $\{\mathcal{M}^{m}_{r}\}^{M}_{m}$ ep e en M o en of e p od c n r e nd e on con e e p d y No ce e ne e co p ed e f nc on e f

e non nd d nd nd d fo

III.1 The Non-Standard Form

Le e n ope o

$$\mathbf{L} \quad \mathbf{R} \to \mathbf{L} \quad \mathbf{R}$$

e e ne y De n n p o ec on ope o on e p ce $\mathbf{V}_j$; $\in \mathbf{Z}$

$$P_j \quad \mathbf{L} \quad \mathbf{R} \to \mathbf{V}_j$$

$$P_j f \quad = \sum_k \langle f \quad {}_{j;k} \rangle \quad {}_{j;k}$$

nd e p nd n n " e e op c e e e o n

$$= \sum_{j \quad \mathbf{Z}} \quad {}_j \quad {}_j \quad {}_j \quad P_j \quad P_j \quad {}_j$$

e e

$$_j = P_j \quad - P_j$$

e p o ec on ope o on e p ce $\mathbf{W}_j$ f e e e co e n en n e d of e e e

$$= \sum_j \quad {}_j \quad {}_j \quad {}_j \quad P_j \quad P_j \quad {}_j \quad P_n \quad P_n$$

nd f e e; = e ne e en

$$= \sum_j \quad {}_j \quad {}_j \quad {}_j \quad P_j \quad P_j \quad {}_j \quad P_n \quad P_n$$

e e $\sim \quad = P \quad P$ d e e z on of e ope o on e ne e p n on e nd deco po e e ope o n o of con on f o d en e e

e non nd d fo ep e en on e of e ope o c n of p e

$$= \{A_j \quad B_j \quad {}_j\}_{j \quad \mathbf{Z}}$$

c n on e p ce $\mathbf{V}_j$ nd $\mathbf{W}_j$

$$A_j \quad \mathbf{W}_j \to \mathbf{W}_j$$

$$B_j \quad \mathbf{V}_j \to \mathbf{W}_j$$

$\mathbf{W_j} \to \mathbf{V_j}$

e e e ope o $\{A_j\ B_j\ \ j\}_{j\ Z}$ e de ned A_j .— j j B_j .— j P_j nd j .—P_j j

e ope o $\{A_j\ B_j\ \ j\}_{j\ Z}$ d ec e de n on e e on

$$j\ .- \begin{pmatrix} A_{j+} & B_{j+} \\ j+ & j+ \end{pmatrix}$$

e e ope o j .—P_j P_j

$$j\ \mathbf{V_j} \to \mathbf{V_j}$$

nd e ope o ep e en ed y e × n pp n

$$\begin{pmatrix} A_{j+} & B_{j+} \\ j+ & j+ \end{pmatrix} \mathbf{W_{j+}} \oplus \mathbf{V_{j+}} \to \mathbf{W_{j+}} \oplus \mathbf{V_{j+}}$$ e

f e e co e n en

$$.- \{\{A_j\ B_j\ \ j\}_{j\ Z\ j\ n\ n}\}$$

e e n .—P_n P_n f e n e of e n e en .— n n nd e ope o e o nzed oc of e ee . e nd

Le e e fo o n o e on

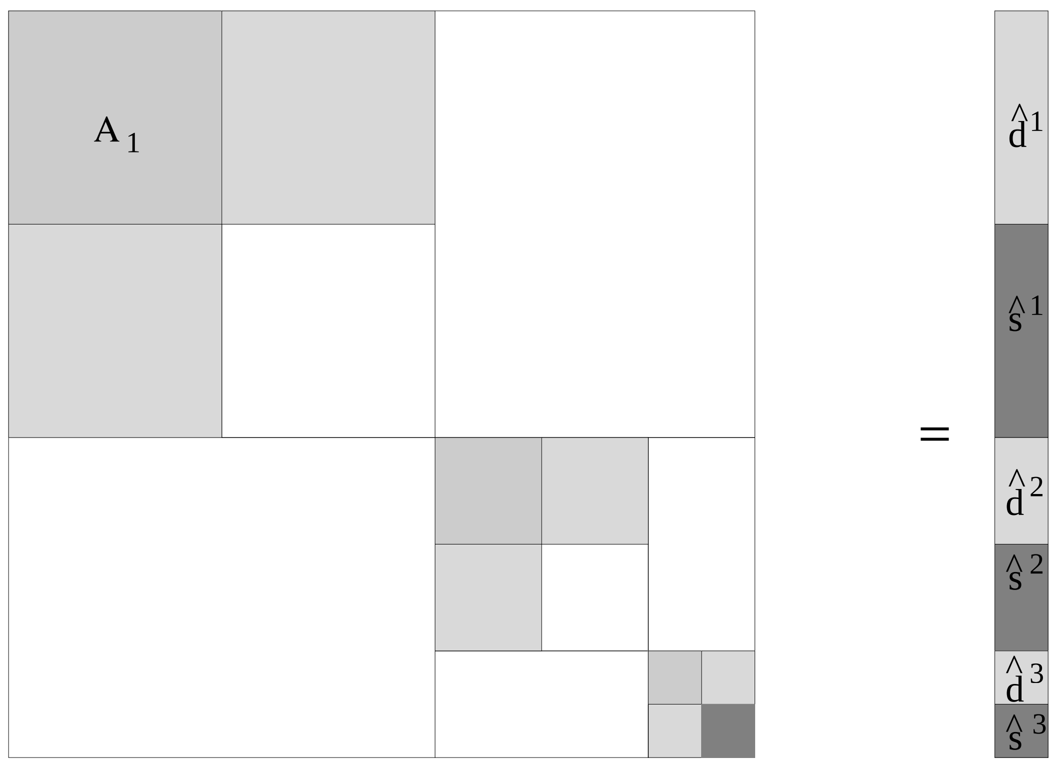
- e ope o A_j de e e n e c on on e e on y nce e p ce $\mathbf{W_j}$ n n e e en of e d ec n
- e ope o B_j j n nd de e e n e c on e een e e nd co e e ndeed e p ce $\mathbf{V_j}$ con n e p ce $\mathbf{V_{j'}}$ ' ee
- e ope o j n " e ed e on of e ope o j
- e ope o A_j B_j nd j e ep e en ed y e ce j j nd j

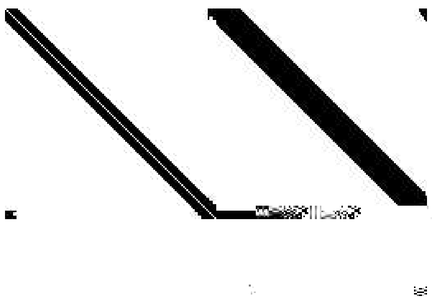
$$^{j}_{k;k'}\ .- \int\int y\ \ _{j;k}\ \ _{j;k'}\ y\ \ d\ dy$$

$$^{j}_{k;k'}\ .- \int\int y\ \ _{j;k}\ \ _{j;k'}\ y\ \ d\ dy \quad 7$$

nd

$$^{j}_{k;k'}\ .- \int\int y\ \ _{j;k}\ \ _{j;k'}\ y\ \ d\ dy$$


A_1
=
$\hat{d}^1$
$\hat{s}^1$
$\hat{d}^2$
$\hat{s}^2$
$\hat{d}^3$
$\hat{s}^3$



e An e p e of n e non nd d fo e p e

e ope o j ep e en ed y e j

$$\mathsf{j}_{\mathsf{k;k'}} = \int\int y \;\; \mathsf{j;k} \;\; \mathsf{j;k'} \; y \;\; d \;\; dy$$

en e of coe c en $\mathsf{k;k'}$ ' = N − epe ed pp c on of e [redacted]
fo 7 p od ce

[redacted]

$$\mathsf{j}_{\mathsf{i;l}} = \sum_{\mathsf{k;m}}^{\mathsf{L}} \mathsf{k} \;\; \mathsf{m} \;\; \mathsf{j}_{\mathsf{k+} \;\; \mathsf{i+} \;\; \mathsf{;m+} \;\; \mathsf{l+}}$$

III.2 The Standard Form

e nd d fo o ned y ep e en n

$$\mathbf{V}_j = \bigoplus_{j'>j} \mathbf{W}_{j'}$$

nd con de n fo e c e e ope o $\{B_j^{j'}, \}_{j'>j}$

$$B_j^{j'} : \mathbf{W}_{j'} \to \mathbf{W}_j$$

$$ {}_j^{j'} : \mathbf{W}_j \to \mathbf{W}_{j'}$$ 7

f e e e co e e n en n e d of e e

$$\mathbf{V}_j = \mathbf{V}_n \bigoplus_{j'=j+}^{n} \mathbf{W}_{j'}$$

n c e e ope o $\{B_j^{j'}, {}_j^{j'}\}$ fo $j' =$ n e e e n nd 7 nd n dd on fo e c e e e ope o $\{B_j^{n+}\}$ nd $\{ {}_j^{n+}\}$

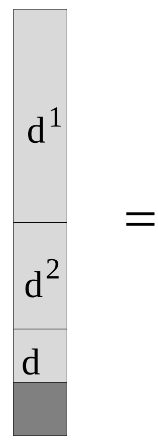
$$B_j^{n+} : \mathbf{V}_n \to \mathbf{W}_j$$

$$ {}_j^{n+} : \mathbf{W}_j \to \mathbf{V}_n$$

n no on ${}_n^{n+} = {}_n$ nd $B_n^{n+} = B_n$ f e n e of e e n e nd $\mathbf{V}$ n e d en on en e nd d fo ep e en on of $= P \; P$

$$= \{A_j \; \{B_j^{j'}\}_{j'}^{n}{}_{j+} \; \{ {}_j^{j'}\}_{j'}^{n}{}_{j+} \; B_j^{n+} \; {}_j^{n+} \; {}_n\}_j$$ re of o C f

e ope o e o n zed oc of e ee e nd e e f e ope o C de on Zy nd o pe do eratoj /R36 (3aV)Tj /R360 Td (Figu65



d1
d2
d
=

e co p e on of ope o

e co p e on of ope o o n o e o d e con c on of e p e ep e
en on n o ono e d ec y ec e peed of co p on o
e e co p e on of d of e fo e p e c e ed y e od o e n
nd n p e ep e en on n e y e deq e fo e pp c on e
co p e on of ope o c fo ep e en on n n o de o e ec e y co
p e n e p e fo e nd d nd non nd d fo of ope o n e e e
e y e e ed co p e on e e e fo de c of nd non nd d fo of ope

the matrices ^{j}, ^{j}, ^{j} (3.16) - (3.18) of the non-standard form satisfy the estimate

$$| _{i;l}^{j}| \quad | _{i;l}^{j}| \quad | _{i;l}^{j}| \leq \frac{C_M}{| - |^{M+}}$$

for all $| - | \geq M$.

con de on pp y n e c e of p e do d en ope o Le e p e do d en ope o y o de ned y e fo

$$f \quad - \int e^{ix} \quad f \quad d - \int \quad y \; f \; y \quad dy$$

e e e d on e ne of

Proposition IV.2 **If the wavelet basis has M vanishing moments, then for any pseudo-di erential operator with symbol of and of satisfying the standard conditions**

$$| \quad _{x} \quad | \leq C \; ; \qquad | \quad +$$

$$| \quad _{x} \quad | \leq C \; ; \qquad | \quad +$$

the matrices ^{j}, ^{j}, ^{j} (3.16) - (3.18) of the non-standard form satisfy the estimate

$$| _{i;l}^{j}| \quad | _{i;l}^{j}| \quad | _{i;l}^{j}| \leq \frac{^{j} \, C_M}{| - |^{\,M+}}$$

for all integer , .

f e pp o e e ope o ^{N} y e ope o $^{N;B}$ o ned f o ^{N} y e n o ze o coe c en of ce $_{i;l}^{j}$ $_{i;l}^{j}$ nd $_{i;l}^{j}$ o de of nd of d $B \geq M$ o nd e d on en e y o ee

$$|| ^{N;B} - ^{N}|| \leq \frac{C}{B^M} \, o \; N$$

e e C con n de e ned y e ene n o n e c pp c on e cc cy of c c on ed nd e p e e of e o n o c e nd d B nd o de M e o e c o en n c nne e de ed p ec on of c c on c e ed f M ed en B o e c

$$|| ^{N;B} - ^{N}|| \leq \frac{C}{B^M} \, o \; N \leq$$

y — y (4.)

e e f c y n y n e f nc on e e nd e nd e efo e e ope o L nd L po e o dec de f C de on Zy nd ope o o nded

Theorem IV.1 (G. David, J.L. Journe) **Suppose that the operator (3.1) satis es the conditions (4.5), (4.6), and (4.16). Then a necessary and su cient condition for to be bounded on L is that in (4.24) and y in (4.25) belong to dyadic BMO, i.e. satisfy condition**

$$\sup_J \frac{1}{|J|} \int_J | \quad - \quad_J | \, d \le C \qquad (4.)$$

where is a dyadic interval and

$$\quad_J = \frac{1}{|J|} \int_J \quad d \qquad (4. 7)$$

p n e ope o n o e of ee e e nd e n e ep e y e d o e e e e e no e e f nc on nd e e y co p ed n e p oce of con c n e non nd d fo nd nd y e ed o p o de f e e of e no of e ope o

e d e en ope o n e e e

V.1 The operator d=dx in wavelet bases

e non nd d fo of e e co on y ed ope o y e co p ed e p c y e con c e non nd d fo of e ope o d d e e e en of A_j B_j nd r of e e e d d d

e e n e e oco e on coe c en of e e $-\{ \}_{k}^{k}$ L

$$n - \sum_{i}^{L} n \quad i \quad i+n \quad n - \quad L -$$

e y o ee e oco e on coe c en n e en nd ce e ze o

$$k - \quad - \quad L -$$

y e e fyed y n o co p e | nd |

$$| - - - \sum_{n}^{L} n \; \mathrm{co}\, n$$

$$| - - - - \sum_{k}^{L=} k \; \mathrm{co} \; - \; - \sum_{k}^{L=} k \; \mathrm{co}$$

e e n e en n Co n n nd o fy e o n

$$\sum_{k}^{L=} k \; \mathrm{co} \; -$$

nd ence nd

e e en o en of e coe c en k f o n n e y

$$\sum_{k}^{k \; L=} k \; - \; m - \quad \text{fo} \quad \leq \; \leq M -$$ e

nce

$$- \; m \; | \; - \quad \text{fo} \quad \leq \; \leq M -$$

c fo o f o e Co o d en f e 7 e d e y fy o 7 7 e f d cof e n — e e e 7

$$r_i := \sum_{k} \sum_{m} k \; k \; m \, r_{i+m}$$

C n n e o de of on n nd n e f c $\sum_{k}^{L} k :=$ e
e

$$r_l := r_l \sum_{n}^{L} n \, r_{l\,n} \, r_{l+n} \quad \in \mathbb{Z}$$

e e n e en n n e o n e f o
n o de o o n e e fo o n e on

$$\sum_{l}^{l} m \; - \; := m \sum_{l}^{l\,m} - l \quad M_l' \; m \; l$$

e e

$$M_l' := \int_{\mathbb{Z}_+} l \, d \quad := $$

e e o en of e f nc on e on fo o p y on n o e
n fo nd n Le n z e n nd — e o n
f $M \geq$ en

$$| \; | \; | \leq C \; |$$

e e nd ence e n e n e y con e en on
fo o f o Le of e e o n

$$| \; | \leq C \; |^{M + \log_2 B}$$

e e

$$B := \sup_{\mathbb{R}} | \; e^{i} \; |$$

D e o e cond on e e o $B := M - \; -$ e
e e ence of on of e e of eq on e nd fo o
f o e e ence of e n e n nce e n f nc on co p c
ppo e e e on y 7 7 d ee e d e

e e

$$r \quad = \sum_{l} r_l e^{il}$$

$$r_{even} \quad = \sum_{l} r \;{}_{l} e^{il} \qquad 7$$

nd

$$r_{odd} \quad = \sum_{l} r \;{}_{l+} e^{i \;\; l+ \;)} =$$

No c n

$$r_{even} \quad = r \quad r$$

nd

$$r_{odd} \quad = r \quad - r \qquad e$$

nd n e o n f o

$$r \quad = r \quad r \qquad \Big| - \quad r \quad - r \qquad e$$

n y n e e

$$r \quad = \quad \Big| r \quad \Big| r \qquad e$$

e n = n e e o n r = r nd e

n q ene of e o on of e nd fo o f o e n q ene of e ep e en on of d d en e on r_l of e nd e con de e ope o $_j$ de ned y e coe c en on e p ce $\mathbf{V}_j$ nd pp y o c en y oo f nc on f nce $r_l^j = {}^j r_l$ e e e

$$_j f \quad = \sum_{k \; Z} {}^j \sum_{l} r_l f_{j;k \;\; l} \quad {}_{j;k} \qquad e$$

e e

$$f_{j;k \;\; l} = {}^{j=} \int^{+} f \qquad {}^j - \qquad d \qquad ee$$

e n ee

$$f_{j;k \;\; l} =$$

d 7 7

e e — j nd | − | ≤ j n e n e nd n e
nd e o n

$$ j f = \sum_{k \in Z} \int f' \;\; j;k \;\; d \;\; j;k $$

$$ j \sum_{k \in Z} \sum_{l} - r_l \int f'' \;\; j;k \;\; d \;\; j;k \;\; ! $$

c e → −∞ ope o j nd d d co nc de on oo f nc on n
nd d en e y o p o e —d d nd ence e o on o e
nd n q e e e on 7 fo o no f o

Remark 2 e no e e p e on nd fo l nd l l — − l y
e p ed y c n n e o de of on n nd nd n od c n
e co e on coe c en P L n i i+n P L n i i+n nd P L n i i+n e
e p e on fo l e pec y p e l —e r l − r l

Examples. o e e p e e D ec e e e con c ed n
e co p e e coe c en k — M e e M en e of n n
o en nd L — M n e on e of

$$ | — − \frac{M -}{M - \quad M} \int_Z n^M \; d $$

e nd y co p n R n M d

$$ | — - C_M \sum_m^M \frac{- m \;\; co \;\; -}{M \quad M \quad - \quad -} $$

e e

$$ C_M — \left[\frac{M -}{M - \quad e^M} \right] $$

y co p n e nd e e

$$ m — \frac{- m \; C_M}{M \quad M \quad - \quad -} \qquad e e — M $$

e no e y e of en o on of ne y e on co
e c en m n e on y con c on e coe c en r_l e on
n e e coe c en r_l e e e fo D ec e e e ed n e
of n n o en M e e e e e e e e fo en

j e d e d en

j ene e
of e o yno e

o n eq on of opo on e p e en e e fo D ec e e e
M . — e

1 M . —

$$. — - \qquad . — - -$$

nd

$$r \; . — - - \qquad r \; . — —$$

e coe c en — — of e p e c n e fo nd n ny oo
on n e c n y c o ce of coe c en fo n e c d en on

---

2 M . —

$$. — \frac{7}{e} \qquad . — - — \qquad . — —$$

nd

$$r \; . — - \frac{7}{\;} \qquad r \; . — — \qquad r \; . — - — \qquad r_4 \; . — - —$$

---

3 M . — e

$$. — \frac{}{e} \qquad . — - \frac{e}{e} \qquad . — \frac{e}{e} \qquad . — - \frac{}{e}$$

nd

$$r \; . — \; - \frac{}{e} \qquad r \; . — \frac{7}{e \; e} \qquad r \; . — - \frac{e}{e}$$

$$r_4 \; . — \; \frac{e \; 7}{\;} \qquad r \; . — \frac{7}{7 \; e} \qquad r \; . — - \frac{7}{\;}$$

---

4 M . —

$$. — \frac{e}{e} \qquad . — - — \qquad . — \frac{7}{\;} \qquad . — - \frac{e \; 7}{\;} \qquad . — \frac{7}{\;}$$

nd

$$r \; . — \; - \frac{7 \quad 7 \quad 7}{e} \qquad r \; . — \frac{7}{e} \qquad r \; . — - \frac{7 \; 7}{\;}$$

$$r_4 \; . — \; \frac{7 \quad 7}{e} \qquad r \; . — -$$

5 M .—

and

Coe cients fo M .— nd M .— c n e co p ed e co e pond n o p f o e fo o n e e o

Iterative algorithm for computing the coe cients r_l.

A y of n eq on e nd e y o n e e o e r .— nd r .—— nd e e n e o eco p e r_l e y o e fy n e nd e ed d e o e c o ce of n z on e fo o n fo D ec e e e M .— co p ed n o d p y e coe c en $\{r_l\}$ e no e r .— $-r_l$ nd r .—

V.2 The operators d^n/dx^n in the wavelet bases

o e ope o d d e non nd d fo of e ope o d^n d ^{n} co p e e y de e ned y ep e en on on e p ce $\mathbf{V}$ e y e coe c en

$$r_l^{(n)} = \int \ldots \frac{d^n}{d^n} \ldots d \qquad \in \mathbf{Z}$$

o e n e y

$$r_l^{(n)} = \int \ldots \xi^n | \ldots \xi | e^{il} d\xi$$

f e n e n o e ee e e o

		Coe cients
	i	$\mid$
M = 5	1	-0.82590601185015
	2	0.22882018706694
	3	-5.3352571932672E-

		Coe cients
	i	$\mid$
M = 8	1	-0.88344604609097
	2	0.30325935147672

Proposition V.2 **1. If the integrals in (5.52) or (5.53) exist, then the coe cients $r_l^{(n)}$, $l \in \mathbf{Z}$ satisfy the following system of linear algebraic equations**

$$r_l^{(n)} = {}^{n} \left[r_l - \sum_{k}^{L=} {}_{k} \; r_{l}^{(n)}{}_{k+} \; r^{(n)}_{l+k} \right] \qquad (\text{e})$$

and

$$\sum_{l} {}^{n} r_l^{(n)} = - {}^{n} n$$

where $_{k}$ **are given in (5.19).**

2. Let $M \geq n$, where M is the number of vanishing moments in (2.16). If the integrals in (5.52) or (5.53) exist, then the equations (5.54) and (5.55) have a unique solution with a nite number of non-zero coe cients $r_l^{(n)}$, namely, $r_l^{(n)} \neq$ for $-L \leq l \leq L -$. Also, for even n

$$r_l^{(n)} = r_{-l}^{(n)}$$

$$\sum_{l} {}^{n} r_l^{(n)} = \quad n = \quad n - \qquad (7)$$

and

$$\sum_{l} r_l^{(n)} =$$

and for odd n

$$r_l^{(n)} = -r_{-l}^{(n)}$$

$$\sum_{l} {}^{n} \; r_l^{(n)} = \quad n = \qquad \leq l \leq L$$

$A \quad e \; M$

e no e on e e e L = e e e e o n n
o en M = do no e e e o de e ponen ee e ep e en on
of e d de e e on y f e n e of n n o en M =

e eq on fo co p n e coe c en $r_l^{n)}$ y e e ed n e en e
p o e Le de e e eq on co e pond n o e fo d^n d ^{n} d ec y f o
e e e

$$r_l^{n)} = \int \sum_{k \in Z} | \quad | ^n \quad ^n e^{il} d$$

e efo e

$$r = \sum_{k \in Z} | \quad | ^n \quad ^n$$

e e

$$r = \sum_l r_l^{n)} e^{il}$$

n e e on

e

n o e nd de of nd n o e e en nd odd nd ce n
ep e y e e

$$r = ^n \quad | r \quad | r$$

Le con de e ope o M on pe od c f nc on d f n f d

$$M f = \quad | f \quad | \quad |$$

n e e e n e c e dence n f c of n e e nce ep
e en one of e d n e of co p n n e e e
e e c e

N	[illegible]	[illegible]p
64	0.14545E+04	0.10792E+02
128	0.58181E+04	0.11511E+02
256	0.23272E+05	0.12091E+02
512	0.93089E+05	

e con o on ope o n e e e

n ec on e conde e co p on of e non nd d fo of con o on ope o . o con o on ope e e q d e fo fo ep e en n e e ne on V e of e p e fo d e

nd e den y fo o f o
o d

nce e o en of e f nc on n eq on e e d o
po n q d e fo fo co p n e ep e en on of con o on
e ne e fo co p c y ppo ed e e fo o
e e nne fo e pec c o ce of e e e de c eqn
e e e f ed o en of e f nc on n e ef p pe fo
e de

e e e n od ce d en pp o c c con n of
ne e c eq on ec o y p o c cond on pe
c y p e f e y o of e ope o o o eneo of p e
c e e ope o co p e e y de ned y e ep e en on on **V**

Le con de o e p e of c ope o e nd
ope o of f c on d en on o n d en on

VI.1 The Hilbert Transform

e pp y o e od o e co p on of e non nd d fo e n
fo

$$\mathcal{H}f\; y \;\; \mathrm{p} \int \frac{f}{\;} d$$

e e p deno e p nc p e

e ep e en on of $\mathcal{H}$ on **V** de ned y e coe c en

$$r_{\mathbf{l}} \;\; \int \;\; \mathcal{H} \;\; d \;\; \in \mathbf{Z}$$

c n n co p e e y de ne o e coe c en of e non nd d
$\mathcal{H} \; \{A_{\mathbf{j}}\; B_{\mathbf{j}}\; \mathbf{j}\}_{\mathbf{j}\in\mathbf{Z}}$ $A_{\mathbf{j}} \; A$ $B_{\mathbf{j}} \; B$ nd $\mathbf{j}$ e e e

	l	Coe cients	l	Coe cients
M = 6	1	-0.588303698	9	-0.035367761
	2	-0.077576414	10	-0.031830988
	3	-0.128743695	11	-0.028937262
	4	-0.075063628	12	-0.026525823
	5	-0.064168018	13	-0.024485376
	6	-0.053041366	14	-0.022736420
	7	-0.045470650	15	-0.021220659
	8	-0.039788641	16	-0.019894368

e e e coe c en r_l of e n fo fo D ec e e e n n o en

e coe c en $r_l \in \mathbf{Z}$ n fy e fo o n e of ne e c eq on

$$r_l - r_l - \sum_{k}^{L} k \; r_{l} \; k + \; r_{l+k}$$

e e e coe c en k e en n n e nd e o n e y p o c of r_l fo e

$$r_l - O \; \overline{M}$$

By e n n e of

$$r_l - \int | \; | \; n \; d$$

e o n $r_l - r_l$ nd r e no e e coe c en r c nno e de e ned f o eq on e nd

o n e e y p o c cond on e co p e e coe c en r_l ny p e ed cc cy

Example.

e co p e e coe c en r_l of e n fo fo D ec e e e n n o en cc cy e coe c en fo e o ned n y p o c e no e $r_l - r_l$ nd r

VI.2 The fractional derivatives

e e fo o n de n on of f c on de e

$$ _{x} f \; = \; \int_{}^{+} \frac{- y \; +}{} f \; y \; dy $$ 7

e e e con de ≠ f en 7 de ne f c on n de e e ep e en on of $_{x}$ on **V** de e ned y e coe c en

$$ r_{l} = \int_{}^{+} \; - \; _{x} \; d \quad \in \mathbf{Z} $$

p o ded n e e

e non nd d fo $_{x} = \{A_{j}\; B_{j}\; , \; _{j}\}_{j \; Z}$ co p ed $A_{j} = \; ^{j}A \quad B_{j} = \; ^{j}B$ nd $_{j} = \; ^{j}$ e e e e en $_{i\;l}$ $_{i\;l}$ nd $_{i\;l}$ of A B nd e o ned f o e coe c en r_{l}

$$ _{i} = \sum_{k}^{L} \sum_{k'}^{L} \; _{k} \; _{k'} \, r \; _{i+k} \; _{k'} $$

$$ _{i} = \sum_{k}^{L} \sum_{k'}^{L} \; _{k} \; _{k'} \, r \; _{i+k} \; _{k'} $$

nd

$$ _{i} = \sum_{k}^{L} \sum_{k'}^{L} \; _{k} \; _{k'} \, r \; _{i+k} \; _{k'} $$

e y o e fy e coe c en r_{l} fy e fo o n y e of ne e c eq on

$$ r_{l} = \left[r_{\;l} \; - \sum_{k}^{L} \; _{k} \; r_{\;l\;k+} \; r_{\;l+\;k} \right] $$

e e e coe c en $_{k}$ e en n n e nd 7 e o n e y p o c of r_{l} fo e

$$ r_{l} \; = \; \frac{}{} \frac{}{+} \; O \frac{}{+ \; + \; M} \quad \text{fo} $$

$$ r_{l} \; = \quad \text{fo} $$ e

Example.

		Coe cients		Coe cients
	l		l	
M = 6	-7	-2.82831017E-06	4	-2.77955293E-02
	-6	-1.68623867E-06	5	-2.61324170E-02
	-5	4.45847796E-04	6	

M p c on of ope o n e e e

VII.1 Multiplication of matrices in the standard form

e p c on of ce of C de on Zy nd nd p e do d e en ope o n e nd d fo eq e o O N o N ope on n dd on po e o con o e d of e n e nd y e n o ze o e en e n e p od c e o e o d of nd e efo e

$$|| \quad - \quad || \leq \qquad (7)$$

e nd de of 7 do n ed y , o e p e f e co p e [4] en e o e one n c n d

VII.2 Multiplication of matrices in the non-standard form

e no o ne n o fo e p c on of e ope o n e non nd d fo ne o e e n e y deco pe e e n e p oce of p c on Le nd e o ope o

$$\mathbf{L} \quad \mathbf{R} \rightarrow \mathbf{L} \quad \mathbf{R} \qquad (77)$$

en e non nd d fo of nd $\{A_j \ B_j \ _j\}_{j \ z}$ nd $\{A_j \ B_j \ _j\}_{j \ z}$ e co p e e non nd d fo $\{A_j \ B_j \ _j\}_{j \ z}$ of $-$

e ec e ope o of

n y e e e 7 of o e

7

e e

$$\sum_{j}^{n} \left[A_j A_j \quad B_j \quad B_j \;\; j \quad A_j B_j \quad j \quad A_j \right] \quad 7$$

nd

$$n \quad n \sum_{j}^{n} P_j \; j B_j P_j \quad 7$$

e ope o n e 7 e c n on fo o n e p ce

$$A_j A_j \quad B_j \; j \quad \mathbf{W}_j \to \mathbf{W}_j \quad 7 \; e$$

$$B_j \;\; j \quad A_j B_j \quad \mathbf{V}_j \to \mathbf{W}_j \quad 7$$

$$j \; j \quad j A_j \quad \mathbf{W}_j \to \mathbf{V}_j \quad 7$$

nd e ope o n e 7

$$j B_j \quad \mathbf{V}_j \to \mathbf{V}_j \quad 7 \; 7$$

e e n 7 d d e f d 7 7 f

of ope on ed e c e e o o e p e e e o n e of
ope on ep p opo on o N
e e n ope o A_j B_j j n tot49 -410.315ac5 0 Td (36su8712.7097 0552

e e o n e e e

e f p c on o of ec on e econd fe o n e of e e o

VIII.1 An iterative algorithm for computing the generalized inverse

n o de o

p oced e nd e e on n e ee n ee n fo e co p on e e pe fo ed on n p c o on nd e ed o ne f o L N AC fo co p n e n e deco po on o e e ed e fo o n f n

$$A_{ij} = \begin{cases} \frac{}{i\,j} & \\ & \end{cases}$$

e e N e cc cy e o d o [4] e en e of X_k e o [4] e e y e c y e o ed f e e c e on

Size $N \times N$	SVD	FWT Generalized Inverse	L_2-Error
128×128	20.27 sec.	25.89 sec.	$3\ 1 \cdot 10^{-4}$
256×256	144.43 sec.	77.98 sec.	$3\ 42 \cdot 10^{-4}$
512×512	1,155 sec. (est.)	242.84 sec.	$6\ 0 \cdot 10^{-4}$
1024×1024	9,244 sec. (est.)	657.09 sec.	$7\ 7 \cdot 10^{-4}$
...	...	...	...
$2^{15} \times 2^{15}$	9.6 years (est.)	1 day (est.)	

Le de e e e e e o nd c n n e c f nc on c c ope o c n e p e en ed e c en y e fo p e do d en op e o N e c e nd e e pe fo nce of e o e epo ed ep e y

VIII.2 An iterative algorithm for computing the projection operator on the null space.

Le con de e fo o n e on

$$X_{k+} = X_k - X_k$$

$$X = A A$$ e

e e A e d o n nd c o en

en $-X_k$ con e e o P_{null} c n e o n e e d ec y o y co n n n n n ep e en on fo $P_{null} := -A\ AA^{\dagger}A$ e e on o co p e e ene zed n e $AA^{\dagger}$ e f p c on o e e e on e f fo de c of ope o e e co p e y e o fo e ene zed n e e po n d ence o e e e doe no eq e co p e y of e n e ope o on y of e po e of e ope o

VIII.3 An iterative algorithm for computing a square root of an operator.

Le de e n e on o con c o A nd A e e A fo p c y e f d o n nd non ne e de n e ope o e con de e fo o n e on

$$Y_{k+} := Y_k - Y_k X_k Y_k$$

$$X_{k+} := -X_k\ Y_k A$$

$$Y := -A$$

$$X := -AA$$

7

e e c o en o e e e en e of $-A$ e n $\sqrt{\ }$ e eq ence X_k con e e o A nd Y_k o A By n $A := V\ DV$ e e D d on nd V n y e y o e fy o X_k nd Y_k c n e en $X_k := V\ P_k V$ nd $Y_k := V\ _k V$ e e P_k nd $_k$ e d on nd

$$_{k+} := \ _k - \ _k P_k \ _k$$

P e e

r n f f f d e y A f d f d

VIII.4 Fast algorithms for computing the exponential, sine and cosine of a matrix

e e ponen of o n ope o e ne nd co ne f nc on e on e o e con de ed n ny c c of ope o A n e c e of e ene zed n e e

X Co p n $F(u)$ n e e e e

n ec on e de e f d p e o fo co p n e e n n n e y d en e f nc on nd ep e en ed n e e An po n e p e O n y c e ene ze eo e of M Bony on e p op on of n e of o on of non ne eq on O n e c pp o c o e e no e e e pec de n e of pp c on of o

IX.1 The algorithm for evaluating u²

e n o o co p e Le p o ec $\in$ **L R** on p ce **V**$_j$ $\in$ **Z** o

$$_j = P_j \quad _j \in \mathbf{V}_j$$

n o de o deco p e e e e e e e op c e e

$$- \; {}_n = \sum_{j}^{n} h \; P_j \quad - \; P_j \quad i = \sum_{j}^{n} P_j \quad P_j \quad P_j \quad - P_j$$

n P_j $= P_j$ $_j$ e o n

$$- \; {}_n = \sum_{j}^{n} P_j \quad _j \quad _j$$

o

$$= \sum_{j}^{n} P_j \quad _j \quad \sum_{j}^{n} \quad _j \quad _j \quad _n$$

n e e e no n e c on e een d en e nd

o e n e c p po e need fo o e n e n e of e o c e

Befo e p oceed n f e e con de n e p e of e n

e e e fo o n e p c e on

$$ j\ k \quad \text{—} \quad j= \quad j\ k $$
$$ j\ k \quad \text{—} \quad j= \quad j\ k \qquad 7 $$
$$ j\ k \quad j\ k \quad \text{—} \quad j= \quad j\ k $$

A o e pod c on e e e e ze o

p nd n e p c y n o

$$ \text{—} \sum_{j}^{n} \sum_{k\,Z} d_k^j \; {}^j_k \qquad \sum_{k\,Z} {}^n_k \; {}^n_k $$

nd n 7 e o n f o e

$$ \text{—} \sum_{j}^{n} \; j= \sum_{k\,Z} d_k^j \; {}^j_k \; {}^j_k \qquad \sum_{j}^{n} \; j= \sum_{k\,Z} d_k^j \; {}^j_k \qquad n= \sum_{k\,Z} {}^n_k \; {}^n_k $$

On deno n

$$ d_k^j \quad \text{—} \quad j= + \; d_k^j \; {}^j_k $$
$$ {}^j_k \quad \text{—} \quad j= \; d_k^j $$
$$ {}^n_k \quad \text{—} \quad n= \; {}^n_k $$

e e e

$$ \text{—} \sum_{j}^{n} \sum_{k\,Z} d_k^j \; {}^j_k \qquad \sum_{j}^{n} \sum_{k\,Z} {}^j_k \; {}^j_k \qquad \sum_{k\,Z} {}^n_k \; {}^n_k $$

e no e **if the coe cient d_k^j is zero then there is no need to keep the corresponding average** ${}^j_k$ n o e o d e need o eep e e on y ne e n e e e e e e e coe c en d_k^j o p od c ${}^j_k d_k^j$ e n c n fo en cc cy

of coe c en c need o e o ed y e ed ced f e y o e n fo e p e

$$M^{\mathbf{j};\mathbf{j}'}_{\mathbf{www}} \quad ' \quad - \quad \mathbf{j}'= \int + \quad \mathbf{j} \quad \mathbf{j}' \quad {}^{\mathbf{j}}_{\mathbf{k}} \ {}^{\mathbf{j}'}_{\mathbf{k}'} \quad {}_{j-j'\mathbf{k}\ \mathbf{l}} \quad d$$

o

$$M^{\mathbf{j};\mathbf{j}'}_{\mathbf{www}} \quad ' \quad - \quad \mathbf{j}'= \ M^{\mathbf{j}\ \ \mathbf{j}'}_{\mathbf{www}} \quad - \quad ' \quad \mathbf{j} \quad \mathbf{j}' \quad -$$

o e e e o n c n ed c on n e n e of coe c en con e q ence of e f c e coe c en n dec y e d nce r $-$ $-$ $'$ e een e e f e n e of n c n coe c en $d_{\mathbf{k}}^{\mathbf{j}}$ p opo on o e n e of c e of N o e e n e of ope on eq ed o e e e pp n

o c e nece y o o e on y o e e $\mathbf{k}'$ c co ne e n c n coe c en $d_{\mathbf{k}}^{\mathbf{j}}$ o p od ce non ze o con on e efo e c en o o e on y o $\mathbf{k}'$ fo c e e e e coe c en $d_{\mathbf{k}}^{\mathbf{j}}$ c $| - '| \leq$ nd e p od c $\mathbf{k}' d_{\mathbf{k}}^{\mathbf{j}}$ o e e e o d of cc cy e n e need o o e e e on y n e ne o ood of n e

e n e of ope on fo e p nd n of e econd e n n o e e e p opo on o e n e of n c n en e nd e e e co p e e y o fo

Remark. e o fo e on n e e e o o e e e p od c of o f nc on nce $\frac{}{4}$ $-$ $-$

IX.2 The algorithm for evaluating $\mathbf{F}(\mathbf{u})$

Le e n n n e y d en e f nc on n o de o deco p e e e e n e e op c e e

$$- \quad \mathbf{n} \quad \sum_{\mathbf{j}}^{\mathbf{n}} \quad P_{\mathbf{j}} \quad - \quad P_{\mathbf{j}} \qquad 7$$

p nd n e f nc on n e y o e e e po n y e P d 7 7 e de p

e no e e e no e e econd de e of n e e e no e en
de e n n nd con de n e e nde of e e e n
n e o e e o n e e of M Bony e e o e o e e e
y oo e n n Bony e nce e e nde f c o j n e d
of j e y eep o e e o e e e nde y oo
no c e po n f e e n d n e n co p n
o epe ed pp c on of e o fo c en o co p e o
f nc on o e e e e e e n y c d n e n con de n n
p c y n eo e c c e z n e

efe ence

. , B A pe p e ep e en on of oo ne ope o D e Y e n e y

. , C e L een d nd o n A f d p e po e o fo p c e on **SIAM Journal of Scienti c and Statistical Computing** e Y e n e y ec n c epo YALeB DO o e

e A Co en D ec e nd C .

M e e of f eq ency c nne deco po on of e nd e e ode ec n c epo e Co n n e of M e c c ence Ne Yo n e y

Y Meye Le c c en q e e onde e e e e o en q d e C MAD n e e D p ne

Y Meye nc pe d nce de e enne e e e d ope e nw6ti3T