



fluid conduits have been observed experimentally, but their properties have never been studied.



scale  $L$  is proportional to the uniform conduit radius while vertical variations are assumed to be weak according to

$$r = r/L, \quad z = \sqrt{1/2} z/L, \quad L = R_0 / \sqrt{\delta}. \quad (18)$$

The proportionality constant in the characteristic length is chosen for convenience in working with the governing equations but will be rescaled to arrive at the standard form of the conduit equation (1). The boundary is now denoted by  $r = (z,t) = R_0 + R(z,t)$  or  $r = (R_0 + R(z,t))/L = \bar{r}(z,t)$ . Hence the unit normal and tangent vectors for the conduit are given by

$$\hat{n}_c = \frac{1}{\bar{r}_c} \begin{pmatrix} \sqrt{1/2} \\ \bar{r}_c \end{pmatrix}, \quad \hat{t}_c = \frac{1}{\bar{t}_c} \begin{pmatrix} 1 \\ \sqrt{1/2} \frac{\bar{r}}{z} \end{pmatrix}, \quad (19a)$$

where

$$\bar{r}_c = \bar{t}_c = 1 + \frac{\bar{r}^2}{z}. \quad (20)$$

Velocities are normalized to the radially averaged vertical velocity of the uniform conduit

$$u^{(i,e)} = u^{(i,e)}/U, \quad U = \frac{gR_0^2 \sqrt{\delta}^{(i)}}{8\mu^{(i)}}, \quad (21)$$

leading to the long time scale  $\sqrt{\delta}^{1/2} T$  for vertical dynamics where

$$t = \sqrt{1/2} t/T, \quad T = L/U. \quad (22)$$

To nondimensionalize the pressure, the characteristic scale is chosen so that the vertical pressure gradient within the conduit balances the viscous force due to radial variation in the vertical velocity,

$$p^{(i,e)} = \sqrt{1/2} \frac{\rho^{(i,e)} \sqrt{\delta} p_0}{\mu^{(i)} U/L}. \quad (23)$$

Like in dimensional variables, the nondimensional, modified pressure can be decomposed as  $p^{(i,e)} = P^{(i,e)} \sqrt{\delta} p_h^{(i,e)}$ , where  $P^{(i,e)} = \sqrt{1/2} p^{(i,e)}/p_0$  is the scaled, absolute pressure and  $p_h^{(i,e)}$  is the normalized hydrostatic pressure which takes the form

$$p_h^{(i,e)} = \sqrt{\delta}^{1/2} \frac{\rho^{(i,e)} g z}{\mu^{(i)} \sqrt{\delta}^{(i)}} = \frac{\sqrt{\delta}^{(i,e)} z}{\mu^{(i)} \sqrt{\delta}^{(i)}}. \quad (24)$$

Surface tension was neglected in the discussion of the uniform conduit, but it will be included in the full system of equations for completeness, so it is normalized about a characteristic scale :

$$\sigma = \sigma / \sigma_c. \quad (25)$$

The Reynolds numbers for the viscous fluid conduit system are therefore defined for the two fluids according to

$$Re^{(i,e)}$$





derive information about higher order corrections in special cases. In what follows, we determine the scalings such that all corrections to the conduit equation (6) are  $O(\epsilon)$ .

#### A. Viscous, higher order corrections

The equations solved in deriving the conduit equation (6) in Sec. III B were a special case of the Stokes- $\tilde{\Omega}$  flow equations, in which the vertical dynamics occurred over a much longer length scale than the radial dynamics. A convenient analytical property of the axisymmetric Stokes- $\tilde{\Omega}$  flow equations, is that one can rewrite the nondimensional equations in the form [23]

$$\nabla^2 p^{(i,e)} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p^{(i,e)}}{\partial r} \right) + \frac{\partial^2 p^{(i,e)}}{\partial z^2} = 0, \quad (58)$$

$$\mathcal{L}^2 \psi^{(i,e)} = 0, \quad \mathcal{L}^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right), \quad (59)$$

where  $\psi^{(i,e)}$  is the Stokes- $\tilde{\Omega}$  stream function, which is related to the velocity components by

$$u_r^{(i,e)} = \tilde{\Omega}^{-1/2} \frac{1}{r} \frac{\partial \psi^{(i,e)}}{\partial z}, \quad u_z^{(i,e)} = \frac{1}{r} \frac{\partial \psi^{(i,e)}}{\partial r}. \quad (60)$$

In the asymptotic formulation, the fluid pressures and velocities were expanded in asymptotic series and expressions for the leading order term in the expansion were found. It was unclear





With the limits of validity of the theoretical model established, viscous fluid conduits provide an optimal environment for the precise, quantitative, experimental study of dispersive shock waves. Dispersively regularized shock waves have attracted a great deal of interest in recent years due to their observation in a wide range of physical systems, to include ultracold, dilute atomic gases [26,27], ion-acoustic plasma [28], nonlinear optics [29,30], and shallow water [31], but careful comparisons of theoretical predictions and data are lacking. One difficulty is the long length scales required for the study of DSWs. These dispersive shock waves are characterized by the presence of two scales. One is the  $\lambda$  scale of individual oscillations and the other is a long, slow scale of wave modulation  $\lambda_m$ . The ratio  $\lambda_m/\lambda$  (generally,  $\lambda_m$  is different from  $\lambda$  defined in Eq. (2)) is the modulation wavenumber. However, images from previous experiments demonstrating the experimental study of DSWs is accessible in viscous fluid conduits, e.g., Fig. 2. In this setting, a DSW is created by a steplike increase in the injection rate. This results in a trailing, vertically uniform conduit connected to a leading, vertically uniform conduit by a regularized dispersive shock wave, as depicted in Fig. 2. By using a computerized syringe pump, high resolution imaging techniques, the measurement of the fluid densities and viscosities, quantitative experiments are possible. With the knowledge of the conduit equation (1), measurements of the asymptotic DSW features (leading and trailing edge amplitudes) can be compared with the results of asymptotic modulation theory [32,33], and the conduit equation (1) by the present author.

FIG. 2. (Color online) Figure 2 shows a vertically uniform, intrusive conduit that is then perturbed by a steplike increase in the injection rate. One can see the formation of a dispersive shock wave (DSW) that slowly modulates the wave train over time. An interesting observation is that the DSW is located within individual waves of the DSW in the fluid.

and Whitehead [5] observed a similar phenomenon when the amplitude of conduit waves exceeded the small slope condition  $1/2R < 1$ . For  $R = 4.5$  and  $2$ , respectively, the evaluation gives  $1/2R = 4.5$  and  $2$ , respectively, indeed the perturbed conduit radius is above the threshold and inertial effects would need to be included to model the conduit dynamics accurately for  $R > 4.5$ . Hence our criteria accurately predict the position of the DSW of the conduit equation as an approximate dispersive shock wave interface.

## CONCLUSIONS

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