Synchronization in large directed networks of coupled phase oscillators

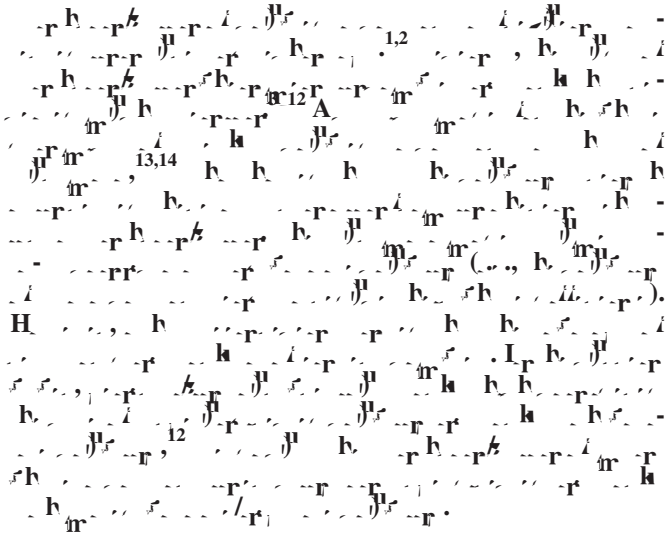
Juan G. Restrepo[a)]
Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, Maryland 20742 and Department of Mathematics, University of Maryland, College Park, Maryland 20742

Edward Ott
Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, Maryland 20742 and Department of Physics and Department of Electrical and Computer Engineering, University of Maryland, College Park, Maryland 20742

Brian R. Hunt
Department of Mathematics, University of Maryland, College Park, Maryland 20742 and Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742

Recei ed 29 A, g 2005; accep ed 11 No ember 2005; p, bli hed online 31 March 2006

We , d he emergence of collec i e nchroni a ion in large direc ed ne ork of he erogeneo, o cilla or b generali ing he cla ical K, ramo o model of globall co, pled pha e o cilla or o more reali ic ne ork . We e end recen heore ical appro ima ion de cribing he ran i ion o nchroni a ion in large , ndirec ed ne ork of co, pled pha e o cilla or o he ca e of direc ed ne ork . We al o con ider he ca e of ne ork i h mi ed po i i e-nega i e co, pling reng h . We compare o, r heor i h n, merical im, la ion and nd good agreemen . – *2005 American Institute of Physics*. DOI: 10.1063/1.2148388



I. INTRODUCTION

The cla ical K, ramo o model[13,14] de cribe a collec ion of globall co, pled pha e o cilla or ha e hibi a ran i ion from incoherence o nchroni a ion a he co, pling reng h i increa ed pa a cri ical al, e. Since real orld ne ork picall ha e a more comple r, c , re han all- o-all co, pling,[15,16] i i na , ral o a k ha effec in erac ion r, c- , re ha on he nchroni a ion ran i ion. In Ref. 12, e , died he K, ramo o model allo ing general connec i i of he node , and fo, nd ha for a large cla of ne ork here i ill a ran i ion o global nchron a he co, pling reng h e ceed a cri ical al, e k_c. We fo, nd ha he cri ical co, pling reng h depend on he large eigen al, e of he adjacenc ma ri A de cribing he ne ork connec i i . We al o de eloped e eral appro ima ion de cribing he beha - ior of an order parame er mea , ring he coherence pa he ran i ion. Thi pa ork a re ric ed o he ca e in hich $A_{nm}=A_{mn}$. 0, ha i , , ndirec ed ne ork in hich he co, - pling end o red, ce he pha e difference of he o cilla or .

Mo ne ork con idered in applica ion are direc ed,[15,16] hich implie an a mme ric adjacenc ma ri , $A_{nm} \neq A_{mn}$. Al o, in ome ca e he co, pling be een o o cilla or migh dri e hem o be o, of pha e, hich can be repre en ed b allo ing he co, pling erm be een he e o - cilla or o be nega i e, A_{nm} 0. The effec ha he pre ence of direc ed and mi ed po i i e-nega i e connec ion can ha e on nchroni a ion i , herefore, of in ere . Here e ho ho o, r pre io, heor can be generali ed o acco, n for he e o fac or . We , d e ample in hich ei her he a mme r of he adjacenc ma ri or he effec of he nega- i e connec ion are par ic, larl e ere and compare o, r he- ore ical appro ima ion i h n, merical ol, ion .

Thi paper i organi ed a follo . In Sec. II e re ie he re , l of Ref. 12 for , ndirec ed ne ork i h po i i e

man he erogeneo, co, pled pha e o cilla or . Thi i , a ion
can be modeled b he eq, a ion

$$\dot{}_n = {}_n + k \sum_{m=1}^{N}$$

A eraging o er he freq, encie , one ob ain he *frequency distribution approximation* FDA

$$r_n = k \sum_m A_{nm} r_m \int_{-1}^{1} g(zkr_m) \sqrt{1 - z^2}\, dz. \qquad (13)$$

The al, e of he cri ical co, pling reng h can be obained from he freq, enc di rib, ion appro ima ion b le ing $r_n \to 0^+$, prod, cing

$$r_n^{(0)} = \frac{k}{k_0} \sum_m A_{nm} r_m^{(0)}, \qquad (14)$$

here $k_0 = 2/$ $g(0)$. The cri ical co, pling reng h h, corre pond o

$$k_c = \frac{k_0}{}, \qquad (15)$$

here i he large eigen al, e of he adjacenc ma ri A and $r^{(0)}$ i propor ional o he corre ponding eigen ec or of A. B con idering per , rba ion from he cri ical al, e a $r_n = r_n^{(0)} + r_n$, e panding $g(zkr_m)$ in Eq. 13 o econd order for mall arg, men , m, l ipl ing Eq. 13 b $r_n^{(0)}$ and ,ming o er n, e ob ained an e pre ion for he order parame er pa he ran i ion alid for ne ork i h rela i el homogeneo, degree di rib, ion [17]

$$r^2 = \left(\frac{1}{k_0^2} \right) \left(\frac{k}{k_c} \right) \left(1 \right) \left(\frac{k}{k_c} \right)^3, \qquad (16)$$

for 0 k/k_c $1 \ll 1$, he2r4115.078hj/F6.768598406.2813Tm2764444111..9.9789j/F599Tc-307.9h37859.di rib, ion 6.913Tm276444

$$r = \prod_{n=1}^{N} r$$

he adjacenc ma ri i independen l cho en o be 1 i h probabili s and 0 i h probabili 1 s, and he diagonal elemen are e o ero. E en ho gh he ne ork con- r c ed in hi a i direc ed, for mo node $d_n^{\text{in}} = d_n^{\text{o}}$. For $N=1500$ and $s=2/15$, Fig. 1a ho he a erage of he order parame er r^2 ob ained from n merical ol ion of Eq. 1 a eraged o er en reali a ion of he ne ork and freq encie riangle , he freq enc di rib ion appro ima- ion FDA, olid line, and he mean eld heor MFT, long da hed line a a f nc ion of k/k_c, here he re l for he FDA and he MFT are a eraged o er he en ne ork real- i a ion no e, ho e er, ha he FDA and he MFT do no depend on he freq enc reali a ion . The per rba ion heor Eq. 16 agreed i h he freq enc di rib ion appro ima ion and a lef o for clari . The error bar cor- re pond o one andard de ia ion of he ample of en real- i a ion . We no e ha he larger error bar occ r af er he ran i ion. When he al e of he order parame er are a er- aged o er en reali a ion of he ne ork and he freq encie , he re l ho er good agreemen i h he freq enc di rib ion appro ima ion and he direc ed mean eld heor .

In order o d ho ell o r heor de cribe ingle reali a ion , e ho in Fig. 1b he order parame er r^2 ob ained from n merical ol ion of Eq. 1 for a par ic lar reali a ion of he ne ork and freq encie bo e , he ime a eraged heor hor da hed line, and he freq enc di - rib ion appro ima ion olid line a a f nc ion of k/k_c. A can be ob er ed from he g re, in con ra i h he ime a eraged heor , he freq enc di rib ion appro ima ion de ia e from he n merical ol ion bo e b a mall b no iceable amo n . Thi beha ior i ob er ed for he o her reali a ion a ell. We no e ha he FDA and MFT re l are ir all iden ical for all en reali a ion . On he o her hand, he TAT and he re l from direc n merical ol ion of Eq. 1 ho dependence on he reali a ion. Since he FDA and MFT incorpora e he reali a ion of he connec- ion A_{nm}, b no he freq encie , e in erpre he ob er ed reali a ion dependence of he TAT and he direc ol ion of Eq. 1 a indica ing ha he la er dependence i d e prima- ril o c a ion in he reali a ion of he freq encie ra her han o c a ion in he reali a ion of A_{nm}.

No e ha for o r e ample $N=1500$ and $s=2/15$ implie ha on a erage e ha e $d^{\text{in}} = d^{\text{o}} = 200$. Th for compari- on p rpo e , e genera ed an ndirec ed ne ork a follo : S ar ing i h=a Eq.a9F54825ek-2 i h TDfrem reali a 4direc

, re , a in he , ndirec ed ca e, he al, e of he a erage of he order parame er ob ained from n, merical ol, ion of Eq. 1. The direc ed per , rba ion heor gi e a good appro i- ma ion for mall al, e of k clo e o k_c, a e pec ed. On he o her hand, he direc ed mean eld heor predic a ran i- ion poin hich i maller han he one ac , all ob er ed.

When n, merically ol ing Eq. (32) b i era ion of Eq. (33), on ome occa ion a period o orbi a fo, nd in ead of he de ired ed poin . If e deno e he lef hand ide of Eq. (33) b z_n^{j+1} and he righ hand ide b $f(z_n^j)$, e fo, nd ha con ergence o a ed poin a facili a ed b replacing he righ hand ide b $(z_n^j + f(z_n^j))/2$ and nding he ed poin of hi modi ed em.

In hi e ample, a lo co, pling reng h (ro, ghl $k/k_c \lesssim 4$, here k_c i comp, ed from Eq. (37)) he order parame er comp, ed from n, merical ol, ion of Eq. (1) i maller han ha ob ained from he TAT and FDA. A k increa e , ho e er, he TAT and FDA heorie cap, re he a mp o ic al, e of he order parame er r. We no e ha in hi ca e he a mp o ic al, e i larger han ha corre ponding o pha e locking (i.e., he one ob ained b e ing $_n=0$ in Eq. (35), $r \approx 0.54 \quad 0.46=0.08$), hich e indica e b a hori on al do -da hed line in Fig. 4, and m, ch maller han $r=1$, he al, e corre ponding o no fr, ra ion (i.e., $_n \quad _m = 0$ for $A_{nm} \not\to 0$ and for $A_{nm} \to 0$ in Eq. (35)). The mall cale of he hori on al a i i d, e o he fac ha e are plo ing r^2, and o o, r de ni ion of he order parame er hich a ign a al, e of 1 o a nonfr, ra ed con g, ra ion. The mall al, e of he order parame er indica e a rong fr, ra ion.

We no e ha in hi e ample, in con ra i h he e - ample di c, ed o far, here i aria ion in he al, e of he order parame er predic ed b he FDA for differen reali a- ion of he ne ork. Thi indica e ha , a he e pec ed al, e of he co, pling reng h A_{nm} become mall (i.e., ($q \quad 1/2$) mall), , c , a ion d, e o he reali a ion of he ne - ork become no iceable. Al ho, gh he al, e predic ed b he FDA and TAT depend on he reali a ion of he ne ork and freq, encie , e no e for $k/k_c \quad 6$ ha he e al, e rack he al, e ob er ed for he n, merical im, la ion of he cor- re ponding reali a ion. A an ill, ra ion of hi , e plo in Fig. 5 he al, e of r^2 ob ained from he TAT (ar) and he al, e of r^2 ob ained from he FDA (diamond) er , he al, e ob ained from n, merical ol, ion of Eq. (1) for $k/k_c = 8$. Each poin corre pond o a gi en reali a ion of he ne - ork, i h re , l a eraged o er en reali a ion of he freq, encie . The ellip e , rro, nding he ar (TAT da a) ha e er ical and hori on al half- id h corre ponding o he an- dard de ia ion of r^2 (TAT) and r^2 (im, la ion) for he en freq, enc reali a ion . The half- id h of he hori on al bar on he diamond (FDA da a) indica e he andard de ia ion of r^2 (im, la ion)

la ion in ne ork i h a m, ch larger n, mber of connec-
ion per node, a he effec of , c , a ion o, ld likel be
red, ced.

of he non ero en rie being cho en randoml e.g., in he mme ric ca e, he po i ion of he non ero en rie i cho en hen con r c ing he ne ork ing he con g ra ion model, and heir al e being al o de ermined randoml from a gi en probabili di rib ion e.g., 1 i h probabili q and 1 i h probabili 1 q. O r in ere i foc ed on he gap be een he large real eigen al e if here i one and he large real par of he o her eigen al e . In Ref. 23 he pec r m of cer ain large par e ma rice i h a erage eigen al e 0 and ro m $\sum_{m=1} A_{nm}=1$ a de cribed and a he ri ic anal ical approach a propo ed. U ing re l for ma rice i h ero mean Ga ian random en rie ,[24] Ref. 23 predic ha he pec r m of he non-Ga ian random ma- rice he con ider con i of a ri ial eigen al e $=1$ i h he remaining eigen al e di rib ed niforml in a circle cen ered a he origin of he comple plane i h radi

$$= \sqrt{N} \ , \qquad \text{(A1)}$$

here ^{2} i he ariance of he en rie of he ma ri . We nd ha hi approach al o cceed in de cribing he pec r m of he ma rice in o r e ample . In o r ca e, he diagonal en rie are 0, o ha he a erage eigen al e i al o 0 a in Ref. 23. We nd ha here i al a a large real eigen al e appro ima el gi en b he mean eld al e

$$= \langle \tilde{d}^2 \rangle / \langle \tilde{d} \rangle \qquad \text{(A2)}$$

ee Ref . 12 and 25, here $\tilde{d}_n = \sum_{m=1}^{N} A_{nm}$ and $\langle \tilde{d}^2 \rangle = \sum_{n=1}^{N} \tilde{d}_n^2$, hich in he ca e con idered in Ref. 23 red ce o $=1$. We al o n mericall con rm ha he remaining eigen al e are niforml di rib ed in a circle of radi a de cribed in Ref. 23. Thi i ill ra ed in Fig. 6.

Th for N if here i a gap of i e be- een he large real eigen al e and real par of he re of he eigen al e pec r m. U ing Eq . (A1) and (A2) i can be ho n ha , for ne ork i h large eno gh n mber of con- nec ion per node or i h eno gh po i i e (or nega i e) bia in he co pling reng h, here i a ide epara ion be een he large eigen al e and he large real par of he remain- ing eigen ec or . For mme ric ma rice , imilar re l ap- pl i.e., he b lk of he pec r m of he ma ri A can be appro ima el ob ained a de cribed abo e ing Wigner' emicircle la .

[1] A. Piko k , M. G. Ro enbl m, and J. K r h , *Synchronization: A univer- sal concept in nonlinear sciences* (Cambridge Uni er i Pre , Cam- bridge, 2001).

[2] E. Mo ekilde, Y. Mai renko, and D. Po no , *Chaotic Synchronization: Applications to Living Systems* (World Scien i c, Singapore, 2002).

[3] L. M. Pecora and T. L. Carroll, Ph . Re . Le . **80**, 2109 (1998).

[4] M. Barahona and L. M. Pecora, Ph . Re . Le . **89**, 054101 (2002).