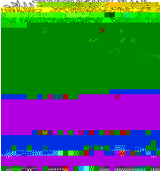


Contents lists available at [ScienceDirect](#)

Journal of Computational Physics

www.elsevier.com/locate/jcp



The functions $u_j^{(l)}(x_j)$ in (1.1) and the corresponding vectors $u_j^{(l)}$ in (1.2) are normalized to have unit Frobenius norm so that the size of the terms is carried by their positive s -values, s_j .

Numerical computations using such representations require an algorithm to reduce the number of terms for a given accuracy, ϵ . Such reduction can be achieved via Alternating Least Squares (ALS) (see, e.g., [35,20,17,13,14,56,39]). Specifically, given a CTD of rank r , ALS allows us to find a representation of the same form but with fewer terms,

$$\tilde{u}_{i_1 \dots i_d} = \sum_{l=1}^k \tilde{u}_{i_1}^{(l)} \tilde{u}_{i_2}^{(l)} \dots \tilde{u}_{i_d}^{(l)}, \quad k < r, \quad (1.3)$$

so that $\| \tilde{u} - \tilde{u}^k \| \leq \epsilon$, where ϵ is a user-selected accuracy. Standard operations on separated representations of rank k , such as multiplication, may result in a large number, e.g., $\mathcal{O}(k^2)$, of intermediate terms. If the intermediate separation rank $r = \mathcal{O}(k^2)$ is reducible to $\mathcal{O}(k)$, then the cost of ALS can be estimated as $\mathcal{O}(d \cdot k^4 \cdot M) \cdot (\text{number of iterations})$, where we assumed that $M_j = M$, $j = 1, \dots, d$. Noting that

$$= \prod_{l=1}^r \mathbf{u}_l^{(l)}, \quad \text{where} \quad \mathbf{u}_j^{(l)} = \prod_{j=1}^d \mathbf{u}_j^{(l)}. \tag{1.5}$$

The standard Frobenius inner product between any two tensors \mathbf{M} and \mathbf{N} is defined as

$$\langle \mathbf{M}, \mathbf{N} \rangle = \prod_{i_1=1}^{M_1} \cdots \prod_{i_d=1}^{M_d} M_{i_1 \dots i_d} N_{i_1 \dots i_d} \tag{1.6}$$

which for CTDs reduces to

$$\langle \mathbf{M}, \mathbf{N} \rangle = \prod_{l=1}^{r_u} \prod_{m=1}^{r_v} \sum_{l=1}^u \sum_{m=1}^v M_{l \dots m}^{(l)} N_{l \dots m}^{(m)} = \prod_{l=1}^{r_u} \prod_{m=1}^{r_v} \sum_{l=1}^u \sum_{m=1}^v \mathbf{u}_j^{(l)} \cdot \mathbf{v}_j^{(m)}, \tag{1.7}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product between component vectors. The Frobenius norm is then defined as

$$\| \mathbf{M} \|_F = \sqrt{\langle \mathbf{M}, \mathbf{M} \rangle}. \tag{1.8}$$

Remark 1.1. The directional vectors $\mathbf{u}_j^{(l)}$ may represent objects of different types, including proper one dimensional vectors, matrices or even low dimensional tensors. The vectors $\mathbf{u}_j^{(l)}$ can have a complicated structure (e.g., sparse

We note that several alternative tensor formats, such as

2.2. Interpolative matrix decomposition

Our approach also relies on computing the interpolative decomposition of a matrix. The idea of matrix ID is to find, for a given accuracy ϵ , a near optimal set of columns (rows) of a matrix so that the rest of columns (rows) can be represented as a linear combination from the selected set. Algorithmically, this decomposition proceeds via pivoted QR factorization and so is

Lemma 2.4. (See Observation 3.3 of [45].) The randomized matrix ID algorithm constructs matrices A_c and P such that

1. some subset of the columns of P makes up the $k \times k$ identity matrix
2. no entry of P has absolute values greater than 2,
3. \langle than

3. Randomized tensor interpolative decomposition

3.1. On the interpolative decomposition of symmetric matrices

We briefly describe a matrix ID for symmetric matrices because it parallels the development for tensors (and, to our knowledge, its description is not available in the literature). Suppose A is an $m \times n$ matrix of rank k (for a given accuracy

Table 1
Estimates of computational complexity.

Reduction method	Computational complexity	
ALS	$(d \cdot k^4 \cdot M) \cdot (\text{number of iterations})$	$(d \cdot k^6) \cdot (\text{number of iterations})$
Tensor ID: random projection	$d \cdot k^3 \cdot M + k^4$	$d \cdot k^5$

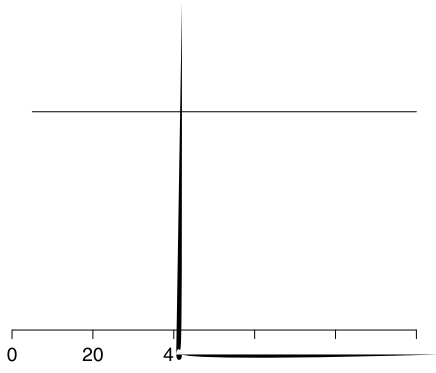
$$\mathbb{E}[\mathcal{R}_{i_1 \dots i_d}] = \mathbb{E} \left[\begin{matrix} d \\ r_{i_j} \\ j=1 \end{matrix} \right] = \begin{matrix} d \\ \mathbb{E}[r_{i_j}] = \mathbf{0}, \\ j=1 \end{matrix} \tag{3.12}$$

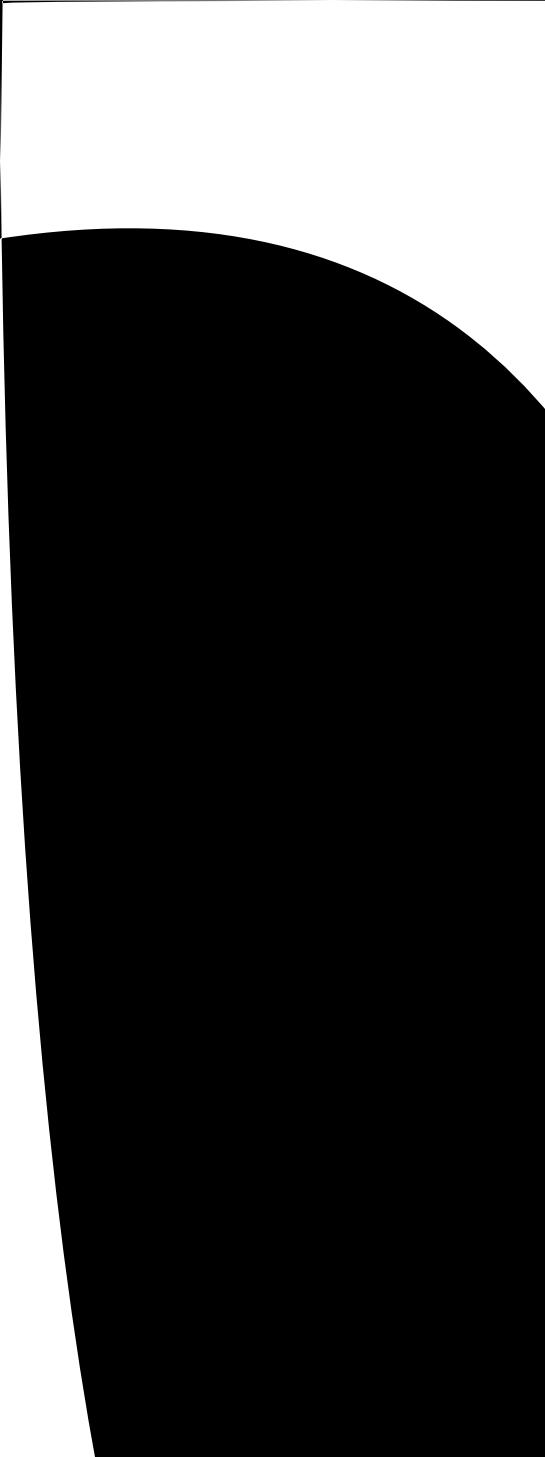
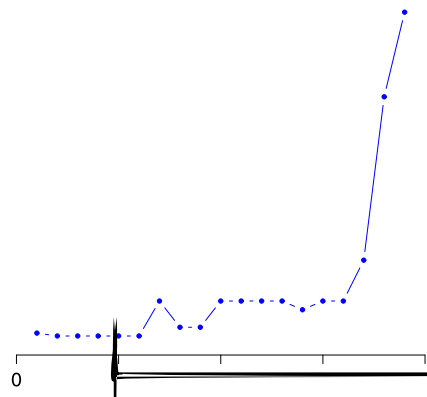
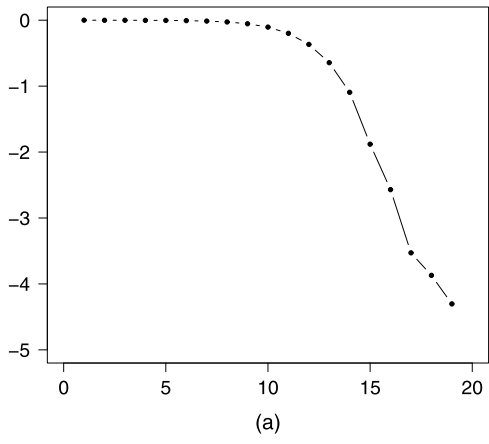
and

$$\mathbb{E} \left[\mathcal{R}_{i_1 \dots i_d} - \mathbb{E}[\mathcal{R}_{i_1 \dots i_d}] \right]^2 = \mathbb{E} \left[\mathcal{R}_{i_1 \dots i_d}^2 \right] = \begin{matrix} d \\ j \end{matrix}$$

Remark 4.2. Although the definition of the *s*-norm parallels the matrix 2-norm and can be useful as a way of estimating errors, computing the *s*-norm exactly for arbitrary dense tensors is claimed to be an NP-hard problem in [36]. We note that for symmetric tensors, the global convergence of the iteration described above has been claimed in [40].

We discuss our approach to initialization below but first consider the cost of estimating the





.....

- [18] E.J. Candès, T. Tao, Near-optimal signal recovery from random projections: universal encoding strategies?, *IEEE Trans. Inf. Theory* 52 (12) (2006) 5406–5425.
- [19] E.J. Candès, J. Romberg, T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, *IEEE Trans. Inf. Theory* 52 (2) (Feb. 2006) 489–509.
- [20] J.D. Carroll, J.J. Chang, Analysis of individual differences in multidimensional trika/T1_2 1 Tf 0 Tc 9.9626 0 0 9.9626 159.4407 606.533 6660(dif /T1_0 1 Tf -