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ON THE REPRESENTATION OF OPERATORS IN BASES OF COMPACTLY SUPPORTED WAVELETS*

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Abstract. This paper describes exact and explicit representations of the di erential operators, n, n = 1 2 ···, in orthonormal bases of compactly supported wavelets as well as the representations of the Hilbert transform and fractional derivatives. The method of computing these representations is directly applicable to multidimensional convolution operators.

Also, sparse representations of shift operators in orthonormal bases of compactly supported wavelets are discussed and a fast algorithm requiring (log) operations for computing the wavelet coe cients of all circulant shifts of a vector of the length = 2^n is constructed. As an example of an application of this algorithm, it is shown that the storage requirements of the fast algorithm for applying the standard form of a pseudodi erential operator to a vector (see [G. Beylkin, R. R. Coifman, and V. Rokhlin, Comm. Pure. Appl. Math., 44 (1991), pp. 141{183]) may be reduced from ( ) to (log^2) signi cant entries.

Key words. wavelets, di erential operators, Hilbert transform, fractional derivatives, pseudodi erential operators, shift operators, numerical algorithms

AMS(MOS) subject classifications. 65D99, 35S99, 65R10, 44A15

1. Introduction.

n D ec e n od ced co p c y ppo ed e e c p o ed o e e y ef n n e c n y n p pe e nd *exact and explicit* ep e en on of e e c ope o de e e n fo f e c n o ono e of co p c y ppo d e e e o p e en n O N o N o fo co p n e e e co cen of N c c n f of ec o of e en N n

o o p pe e on y co p e e non nd d fo of ope o nce p e e o o n nd d fo f o e non nd d fo Meye fo o n fon d f d of c d of fc n de op d oo f d co p d n n d fon fo econd e co p e e non nd d fo of e f ope o ope o po n n p c c pp c on of e e ec e e e e co c en e no f n n nce e non nd d nd nd d fo of ope o e p e nd e y o co p e no n e e ep e en on co pen e fo e c of f n nce e e e e p n on of f of o o of ce y e o ned y p y n e f ope o d ec y o e co c en of e o n e p n on e co c en fo e f ope o y e o ed n d nce nd ed needed

ce o e e e p c nne n c p ene of e f ope o y e e p o ed depend on e pp c on nd y e e fo d n nd c ed o e e p e en n e pe of c n pp c on n n e c n y O e n e e e on y **N** o 2 **N** *distinct* e e co c en n e deco po on of **N** c c n f of ec o of e en **N** e con c n **O N** o **N** o fo co p n of e e co c en n o e o e o e eq e en of e f o fo pp y n e nd d fo of p e d d e en ope o o ec o y e ed ced f o **O N** o **N** o **O** o 2 **N** n c n en e

2. Compactly supported wavelets.

n ec on e e y e e e o ono e of co p c y ppo ed e e nd e o no on o e de e efe o

e o ono of co p c y ppo ed e e of $L^2(R)$ fo ed y e d on nd n on of n e f nc on (x)

$$_{j,k}(x) = \quad ^{-j/2} \quad ^{-j}x \quad k) ;$$

e e $j;k \in Z$ e f nc on (x) co p n on e c n f nc on $'(x)$ nd e e f nc on fy e fo o n e on

$$'(x) = \sqrt{\ } \sum_{k=0}^{L-1} h_k\, ' \quad x \quad k) ;$$

$$(x) = \sqrt{\ } \sum_{k=0}^{L-1} g_k\, ' \quad x \quad k) ;$$

e e

$$g_k = \quad ^k h_{L-k-1}; \quad k = \quad ; \quad ; L \quad ;$$

nd

$$\int_{-\infty}^{+\infty} '(x)\, dx = \quad :$$

n dd on e f nc on M n n o en

$$\int_{-\infty}^{+\infty} (x)\, x^m dx = \quad ; \quad m = \quad ; \quad ; M \quad :$$

where

$$\mathbf{P}\ \mathbf{y} \quad \sum_{k=0}^{k=M-1} \quad \mathbf{M} \quad \mathbf{k} \quad \mathbf{k} \quad ! \quad \mathbf{y}^{k};$$

and **R** *is an odd polynomial such that*

$$\mathbf{P}\ \mathbf{y} \quad \mathbf{y}^{M}\mathbf{R}\ \tfrac{1}{2} \quad \mathbf{y} \quad \text{fo} \quad \mathbf{y} \quad ;$$

and

$$\underset{0\le y\le 1}{\text{p}} \ \mathbf{P}\ \mathbf{y} \quad \mathbf{y}^{M}\mathbf{R}\ \tfrac{1}{2} \quad \mathbf{y} \ < \ ^{2(M-1)}:$$

3. The operator d=dx in wavelet bases. n ec on e con c e non nd d fo of e ope o **d=dx** e non nd d fo ep e en on of n ope o **T** c n of p e

$$\mathbf{T} \quad \mathbf{f}\mathbf{A}_{j};\mathbf{B}_{j}; \ _{j}\mathbf{g}_{j\in\mathbf{Z}}$$

c n on e p ce $\mathbf{V}_j$ nd $\mathbf{W}_j$

$$\mathbf{A}_{j} \ \mathbf{W}_{j} \ ! \ \mathbf{W}_{j};$$

$$\mathbf{B}_{j} \ \mathbf{V}_{j} \ ! \ \mathbf{W}_{j};$$

$$_{j} \ \mathbf{W}_{j} \ ! \ \mathbf{V}_{j}:$$

e ope o $\mathbf{f}\mathbf{A}_{j};\mathbf{B}_{j}; \ _{j}\mathbf{g}_{j\in\mathbf{Z}}$ e de ned $\mathbf{A}_{j}$ $\mathbf{Q}_{j}\mathbf{T}\mathbf{Q}_{j}$ $\mathbf{B}_{j}$ $\mathbf{Q}_{j}\mathbf{T}\mathbf{P}_{j}$ nd $_{j}$ $\mathbf{P}_{j}\mathbf{T}\mathbf{Q}_{j}$ e e $\mathbf{P}_{j}$ e p o ec on ope o on e p ce $\mathbf{V}_j$ nd $\mathbf{Q}_{j}$ $\mathbf{P}_{j-1}$ $\mathbf{P}_{j}$ e p o ec on ope o on e p ce $\mathbf{W}_j$

e e en $^{j}_{il}$ $^{j}_{il}$ $^{j}_{il}$ of $\mathbf{A}_{j}$ $\mathbf{B}_{j}$ $_{j}$ nd $\mathbf{r}^{j}_{il}$ of $\mathbf{T}_{j}$ $\mathbf{P}_{j}\mathbf{T}\mathbf{P}_{j}$ $\mathbf{i};\mathbf{l};\mathbf{j}\ 2\ \mathbf{Z}$ fo e ope o **d=dx**

o p o e e on e co p e $|m_0(\xi)|^2$ n nd o n

$$|m_0(\xi)|^2 = \ldots - \ldots \sum_{n=1}^{L-1} a_n \cos(n\xi);$$

e e a_n e en n Co p n $|m_0(\xi)|^2$; e e

$$|m_0(\xi)|^2 \ldots - \ldots \sum_{k=1}^{L/2} a_{2k-1} \cos(\ldots k \ldots) - \sum_{k=1}^{L/2-1} a_{2k} \cos(\ldots k):$$

Co n n nd o fy e o n

$$\sum_{k=1}^{L/2-1} a_{2k} \cos(k \ldots) \ldots;$$

nd ence nd ee o e o n n o en of a_{2k-1}

e p o e e fo o n

P OPO ON

If the integr

e n n e o n **r** **r** nd
n q ene of e o on of nd fo o f o$_n$ e n q ene of
e ep e en on of **d=dx** en e o on $\mathbf{r}_l$ of nd

OPERATORS IN

M

a_1

OPERATORS IN

P OPO ON

If the integrals in or exist, then the coefficients $r_l^{(n)}$; $l \in Z$ *satisfy the following system of linear algebraic equations*

$$r_l^{(n)} \quad ^n \left[r_{2l} \quad - \sum_{k=1}^{L/2} a_{2k-1} \; r_{2l-2k+1}^{(n)} \quad r_{2l+2k-1}^{(n)} \right];$$

and

$$\sum_l l^n \, r_l^{(n)} \quad ^n \, n \, ;$$

where a_{2k-1} *are given in* .

Let M n $=$; *where* M *is the number of vanishing moments in* . *If the integrals in or exist; then the equations and have a unique solution with a finite number of nonzero coefficients* $r_l^{(n)}$; *namely;* $r_l^{(n)} \neq$ *for* $L \quad l \quad L$; *such that for even* n

$$r_l^{(n)} \quad r_{-l}^{(n)};$$

$$\sum_l l^{2\tilde{n}} \, r_l^{(n)} \quad ; \quad \tilde{n} \quad ; \quad ; n = \quad ;$$

and

$$\sum_l r_l^{(n)} \quad ;$$

and for odd n

$$r_l^{(n)} \quad r_{-l}^{(n)};$$

$$\sum_l l^{2\tilde{n}-1} \, r_l^{(n)} \quad ; \quad \tilde{n} \quad ; \quad ; \; n \quad = :$$

e p oof of P opo on co p e e y n o o o of P opo on

Remark e ne y e n P opo on y e n q e o on e e n e nd e no o e y con e en A c e n po n e D ec e e e M e ep e en on of e de e n de c ed n e p e o ec on Eq on nd do no e o on fo e econd de e n o e e e y e of eq on nd o on fo e d de e n e e

$$a_1 \quad -; \qquad a_3 \quad -;$$

nd

$$r_{-2} \quad -; \quad r_{-1} \quad ; \quad r_0 \quad ; \quad r_1 \quad ; \quad r_2 \quad -:$$

e e of o c en = ; ; ; ; = one of e nd d c o ce of n e d e ence co c en fo e d de e

e no e on e e e **L** e e e o n n o en **M** do no e e e o de e ponen ee e ep e en on of e d de e e on y f e n e of n n o en **M**

Remark Le de e n eq on ene z n o fo d^n/dx^n d ec y f o e e e

$$r_l^{(n)} \quad \int_0^{2\pi} \sum_{k\in\mathbf{Z}} |\varphi \quad k|^2 \quad n \quad k^n e^{-il\xi} d :$$

e efo e

$$r \quad \sum_{k\in\mathbf{Z}} |\varphi \quad k|^2 \quad n \quad k^n;$$

e e

$$r \quad \sum_l r_l^{(n)} e^{il\xi}.$$

n e e on

$$\varphi \quad m_0 = \varphi =$$

n o e nd de of nd n ep e y o e e en nd odd nd ce n e e

$$r \quad n\, |m_0 = |^2 r = |m_0 = |^2 r = :$$

By con de n e ope o M_0 de ned on pe od c f nc on

$$M_0 f \quad |m_0 = |^2 f = |m_0 = |^2 f = ;$$

e e e

$$M_0 r \quad {}^{-n} r.$$

r n e en ec o of e ope o M_0 co e pond n o e e en e $^{-n}$ nd e efo e nd n e ep e en on of e de e n e e e eq en o nd n ono e c po yno o on of nd ce e e ope o M_0 o n od ced n nd e e e p o e e en e con de ed

Remark e eo e c y e nde ood e de e ope o o e ene y ope o o eneo y o e n e p c d on p econd one n e e e en e c e dence n f c of n e e nce ep e en one of e d n e of co p n n e e e e f n ope o n p ce e c n p ce o n p ce fo e c c cy en y e cond on n e e nde nd e o of e e n e o e e n e *above* e e o d of cc cy e nc de e on e e e ope o y e p econd oned on y on p ce e no e e p econd on n de c ed e e dd e e e p o e of cond on n d e on y o e nf o e o o ene y of e y o nd doe no ec cond on n d e o o e c e

o pe od zed de e ope o e o nd on e cond on n e depend on y on e p c c o ce of e e e Af e pp y n c p econd one e cond on n e $_p$ of e ope o n fo y o nded e pec o e ze of e e ec e cond on n e con o e e of con e ence of n e of e e o fo e p e e n e of e on of e con e d en e od O p — $_p$ e p e co p e e y ne o oo on n e of n e c e od op c e dd e e e e e e p e en e e o e n c p econd on n pp ed o e n d d fo of e econd de e ee on o o co p e e nd d fo f o e non nd d fo n e fo o n e p e e nd d fo of e pe od zed econd de e D_2 of ze N N e e N ^{n} p econd oned y e d on P

$$D_2^p \quad P D_2 P$$

e e P_{il} $_{il}$ ^{j} j n nd e e j c o en depend n on i;l o N N= $^{j-1}$ i;l N N= ^{j} nd P_{NN} ^{n}

e nd co p e e o n cond on n e

Table 3

Condition numbers of the matrix of periodized second derivative (with and without preconditioning

n nd e den y ' ' = $\mathbf{m}_0$ = ee ce o d p o ded

$$-\partial_\xi^{m} \; |\mathbf{m}_0 \; |^2 \big|_{\xi=0} \quad \text{fo} \quad \mathbf{m} \quad \mathbf{M} \; ;$$

o d e o

$$-\partial_\xi^{m} \; |\mathbf{m}_0 \; |^2 \big|_{\xi=0} \quad \text{fo} \quad \mathbf{m} \quad \mathbf{M} \; :$$

B fo fo o f o e e p c ep e en on n

Remark Eq on nd o py e en o en of e co c en $\mathbf{a}_{2k-1}$ f o n n e y

$$\sum_{k=1}^{k=L/2} \mathbf{a}_{2k-1} \; \mathbf{k}^{2m} \quad \text{fo} \quad \mathbf{m} \quad \mathbf{M} \; :$$

nce e o en of e f nc on n eq on e d o one po n q d e fo fo co p n e ep e en on of con o on ope o on e ne c e fo o ned n e c y e e nne fo e pec c o ce of e e e de c ed n eqn e e e f ed o en of e f nc on n e efe o p pe fo e de

e e e n od ce d e en pp o c fo co p n ep e en on of con o on ope o n e e e c con of o n e y e of ne e c eq on ec o y p o c cond on e od e pec y p e f e y o of e ope o n ceneo B onfe d eq on e

: due

nd

$$i \quad \sum_{k=0}^{L-1} \sum_{k=0}^{L-1} \mathbf{h}_k \mathbf{g}_k \; \mathbf{r}_{2i+k-k} \; :$$

e co c en $\mathbf{r}_l$ l 2 Z n fy e fo o n y e of ne e c eq on

$$\mathbf{r}_l \quad \mathbf{r}_{2l} \quad - \quad \sum_{k=1}^{L/2} \mathbf{a}_{2k-1} \; \mathbf{r}_{2l-2k+1} \quad \mathbf{r}_{2l+2k-1} \; ;$$

e e e co c en $\mathbf{a}_{2k-1}$ e en n n nd e o n e y p o c of $\mathbf{r}_l$ fo e l

$$\mathbf{r}_l \quad \mathbf{l} \quad O \quad \mathbf{l}^{2M} \quad :$$

By e n n e of '

$$\mathbf{r}_l \quad \int_0^\infty \mathbf{j}' \quad \mathbf{j}^2 \; \mathrm{n} \; \mathbf{l} \quad \mathbf{d} \; :$$

e o n $\mathbf{r}_l$ $\mathbf{r}_{-l}$ nd e $\mathbf{r}_0$ e no e e co c en $\mathbf{r}_0$ c nno e de e ned f o eq on nd

o n e y p o c cond on e co p e e co c en $\mathbf{r}_l$ l 6 ny p e c ed cc cy e no e e ene z on fo co p n e co c en of e z n fo n e en on fo d

Example e co p e ee e e co c en $\mathbf{r}_l$ of e e n fo fo D ec e e e n n o en e e d

Examp

Table 5

The coefficients $\{r_l\}_l$, $l = -7 \cdots 14$ of the fractional derivative $\alpha = 0.5$ for Daubechies' wavelet with six vanishing moments.

	l	Coefficients r_l	l	Coefficients r_l
M = 6	-7	-2.82831017E-06	4	-2.77955293E-02
	-6	-1.68623867E-06	5	-2.61324170E-02
	-5	4.45847796E-04	6	-1.91718816E-02
	-4	-4.34633415E-03	7	-1.52272841E-02
	-3	2.28821728E-02	8	-1.24667403E-02
	-2	-8.49883759E-02	9	-1.04479500E-02
	-1	0.27799963	10	-8.92061945E-03
	0	0.84681966	11	-7.73225246E-03
	1	-0.69847577	12	-6.78614593E-03
	2	2.36400139E-02	13	-6.01838599E-03
	3	-8.97463780E-02	14	-5.38521459E-03

6. Shift operator on $\mathbf{V}_0$ and fast wavelet decomposition of all circulant shifts of a vector. Le con de f y one on e p ce $\mathbf{V}_0$ ep e en ed y e

$$\mathbf{t}^{(0)}_{i-j} \quad {}_{i-j,1};$$

e e e onec e y n e $\mathbf{a}_n$ of e e

$$\mathbf{t}^{(0)}_{l} \quad {}_{l,1}; \qquad \mathbf{t}^{(1)}_{l} \quad \tfrac{1}{2}\mathbf{a}_{|2l-1|}; \qquad :$$

e on y nonze o co c en $\mathbf{t}^{(j)}_l$ on e c c e **j** e o e nd ce **L** **l** **L** A $\mathbf{t}^{(j)}_l$! ${}_{l,0}$ **j** ! **1** A n e e e fo o n e con n e co c en $\mathbf{t}^{(j)}_l$ **j** ; ; ; fo e f ope o n D c e e e ee n n n

e no e e f y n n e o e n one e ed y o e e f e o e e of e f e e n **L** en on e e e c e **j** e e e nonze o co en $\mathbf{t}^{(j)}_l$ **l** o de e n e **jlj** **L** A **j** nc e e e nonze o co c en $\mathbf{t}^{(j)}_l$ e nd ce n e n e **lj** **L**

e po nce of e f ope o e f o e f c e co c en of e e n fo e no f n n o e e e e de n ed e non nd d nd e fo e e nd d fo of e f ope o e p e nd e y o co e P c en

Table 6

The coefficients $\{ {}^{(j)}_{l} \}_{l=-L+2}^{l=L-2}$ for Daubechies' wavelet with three vanishing moments, where $L = 6$ and $j = 1, \cdots, 8$.

	l	Coefficients ${}^{(j)}_{l}$		l	Coefficients ${}^{(j)}_{l}$
j = 1	-4	0.	j = 5	-4	-8.3516169979703E-06
	-3	0.		-3	-4.0407157939626E-04
	-2	1.171875E-02		-2	4.1333660119562E-03
	-1	-9.765625E-02		-1	-2.1698923046642E-02
	0	0.5859375		0	0.99752855458064
	1	0.5859375		1	2.4860978555807E-02
	2	-9.765625E-02		2	-4.9328931709169E-03
	3	1.171875E-02		3	5.0836550508393E-04
	4	0.		4	1.2974760466022E-05
j = 2	-4	0.	j = 6	-4	-4.7352138210499E-06
	-3	-1.1444091796875E-03		-3	-2.1482413927743E-04
	-2	1.6403198242188E-02		-2	2.1652627381741E-03
	-1	-1.0258483886719E-01		-1	-1.1239479930566E-02
	0	0.87089538574219		0	0.99937113652686
	1	0.26206970214844		1	1.2046257104714E-02
	2	-5.1498413085938E-02		2	-2.3712690179423E-03
	3	5.7220458984375E-03		3	2.4169452359502E-04
	4	1.3732910156250E-04		4	5.9574082627023E-06
j = 3	-4	-1.3411045074463E-05	j = 7	-4	-2.5174703821573E-06
	-3	-1.0904073715210E-03		-3	-1.1073373558501E-04
	-2	1.2418627738953E-02		-2	1.1081638044863E-03
	-1	-6.9901347160339E-02		-1	-5.7198034904338E-03
	0	0.96389651298523		0	0.99984123346637
	1	0.11541545391083		1	5.9237906308573E-03
	2	-2.3304820060730E-02		2	-1.1605296576369E-03
	3	2.5123357772827E-03		3	1.1756409462604E-04
	4	6.7055225372314E-05		4	2.8323576983791E-06

j = 4	-4	-1.2778211385012E-05	j = 8	-4	-1.2976609638869E-06
	-3	-7.1267131716013E-04		-3	-5.6215105787797E-05
	-2	7.5265066698194E-03		-2	5.6059346249153E-04
	-1	-4.0419702418149E-02		-1	-2.8852840759448E-03
	0	0.99042607471347		0	0.99996009015421
	1	5.2607019431889E-02		1	2.9366035254748E-03
	2	-1.0551069863141E-02		2	-5.7380655655486E-04
	3	1.1071795597672E-03		3	5.7938552839535E-05
	4	2.9441434890032E-05		4	1.3777042338989E-06

ope o o o pe p c e n e co p e ed fo e co c en
$\mathbf{t}_l^{(j)}$ fo e f ope o c n e o ed n d nce nd ed needed c e
o e e e od of n p ene of e f ope o depend on e
pec c pp c on nd y e e fo d n nd c ed o e
e fo o n n e p e of n pp c on e e n e d of co p n f
ope o e co p e po e f e de c e f o fo e e e
deco po on of c c n f of ec o nd en o o y e ed o
ed ce o e eq e en of one of e o of
e ec e deco po on of e o of en **N** ^{n} n o e e
eq e **O N** ope on nce e co c en e no f n n e

fo o e $\mathbf{s}_k^{j-1}$ k ; ; $n-j$ e one of e ec o of e e on e p e o c e j nd co p e

$$\mathbf{s}_k^j = \sum_{n=0}^{n=L-1} \mathbf{h}_n \mathbf{s}_{n+2k-1}^{j-1};$$

$$\mathbf{s}_k^j = \sum_{n=0}^{n=L-1} \mathbf{h}_n \mathbf{s}_{n+2k}^{j-1};$$

nd

$$\mathbf{d}_k^j = \sum_{n=0}^{n=L-1} \mathbf{g}_n \mathbf{s}_{n+2k-1}^{j-1};$$

$$\mathbf{d}_k^j = \sum_{n=0}^{n=L-1} \mathbf{g}_n \mathbf{s}_{n+2k}^{j-1}:$$

o co p e e e n nd e f y one e eq ence $\mathbf{s}_k^{j-1}$ n nd

epp n f o c e o c e e do e e n e of ec o of e e nd of d e ence nd e e e e e en of e c of e e efo e e o n e of ope on n co p on O N o N

Le o n ze e ec o of d e ence nd e e fo o on e c e j e e

$$\mathbf{v}_1 \quad \mathbf{d}_k^1 \quad ; \mathbf{d}_k^1$$

nd

$$\mathbf{u}_1 \quad \mathbf{s}_k^1 \quad ; \mathbf{s}_k^1 \quad ;$$

e e $\mathbf{d}_k^1$ $\mathbf{d}_k^1$ $\mathbf{s}_k^1$ nd $\mathbf{s}_k^1$ e co p ed f o $\mathbf{s}_k^0$ cco d n o

On e econd c e j e e

$$\mathbf{v}_2 \quad \mathbf{d}_k^2 \quad ; \mathbf{d}$$ e ge 8)

ne c e fo s^1_k nd s^1_k e n o n po e co c en fo odd nd e en f c e co ec n $\mathbf{v}_2$ nd $\mathbf{u}_2$ e c e e ec o $\mathbf{v}_1; \mathbf{v}_2; \ldots; \mathbf{v}_n$ con n e co c en e e co c en e no o n zed eq en y n o de o cce e ene e o e $\mathbf{i}_{\mathrm{loc}}(\mathbf{i}_s; \mathbf{j})$ nd $\mathbf{i}_b(\mathbf{i}_s; \mathbf{j})$ n $\mathbf{O}(\mathbf{N})$ o $\mathbf{N}$ ope on fo o o e c f $\mathbf{i}_s$ $\mathbf{i}_s$ $\mathbf{N}$ of e ec o $\mathbf{s}^0_k; \mathbf{k}$; ; ; $\mathbf{N}$ e e e n y e p n on of $\mathbf{i}_s$

$$\mathbf{i}_s = \sum_{l=0}^{l=n-1} \; l \; {}^l;$$

e e l ; o ed c e $\mathbf{j}$ $\mathbf{j}$ $\mathbf{n}$ e co p e

$$\mathbf{i}_{loc}(\mathbf{i}_s; \mathbf{j}) = \sum_{l=0}^{l=j-1} \; l \; {}^l;$$

nd

$$\mathbf{i}_b(\mathbf{i}_s; \mathbf{j}) = \sum_{l=n-1}^{l=j} \; l \; {}^l;$$

e e $\mathbf{i}_b(\mathbf{i}_s; \mathbf{j})$ f $\mathbf{j}$ $\mathbf{n}$ e n e $\mathbf{i}_b(\mathbf{i}_s; \mathbf{j})$ po n o e e n n of e ec o of d e ence n $\mathbf{v}_j$ N e y e ec o of $\mathbf{v}_j$ nd ce e een $\mathbf{i}_b(\mathbf{i}_s; \mathbf{j})$ nd $\mathbf{i}_b(\mathbf{i}_s; \mathbf{j})$ $^{n-j}$ n ec o c e ed pe od c ec o e pe od $^{n-j}$ e n e $\mathbf{i}_{\mathrm{loc}}(\mathbf{i}_s; \mathbf{j})$ po n o e e e en o c e $\mathbf{j}$ $\mathbf{j}$ $\mathbf{n}$ nd f $\mathbf{i}_s$ $\mathbf{i}_s$ $\mathbf{N}$ e co p e o e n nd e e e e e d ec cce o e co c en n ec o $\mathbf{v}_1; \mathbf{v}_2; \ldots; \mathbf{v}_n$ fo con n co pe e e en

e no e y de c e one of e pp c on of e o fo e f e e deco po on of c c n f of ec o n e c n y e o of e de ned o e e e C de on Zy nd o p e dod e en ope o $\mathbf{T}$ e ne $\mathbf{K}(\mathbf{x}; \mathbf{y})$

$$\mathbf{g}(\mathbf{x}) = \int_{-\infty}^{+\infty} \mathbf{K}(\mathbf{x}; \mathbf{y}) \, \mathbf{f}(\mathbf{y}) \, \mathbf{dy}$$

y con c n fo ny ed cc cy p e non nd d o nd d fo nd e e y ed c n e co of pp y n o f nc on

Le e e

$$\mathbf{g}(\mathbf{x}) = \int_{-\infty}^{+\infty} \mathbf{K}(\mathbf{x}; \mathbf{x} - \mathbf{z}) \, \mathbf{f}(\mathbf{x} - \mathbf{z}) \, \mathbf{dz}:$$

f e ope o $\mathbf{T}$ con o on en $\mathbf{K}(\mathbf{x}; \mathbf{x} - \mathbf{z}) = \mathbf{K}(\mathbf{z})$ f nc on of $\mathbf{z}$ on y e non nd d fo of con o on eq e p $\mathbf{O}$ o $\mathbf{N}$ of o e ee e pe o ec on e e nd d fo of con n $\mathbf{O}(\mathbf{N})$ o $\mathbf{O}(\mathbf{N})$ o $\mathbf{N}$ n c n en e e en fo con o on A e n e y e nd d fo of $\mathbf{K}(\mathbf{x}; \mathbf{x} - \mathbf{z}) = \mathbf{K}(\mathbf{z})$ n e $\mathbf{x}$ nd $\mathbf{z}$ fo e con o on ope o con n no p e n $\mathbf{O}$ o $\mathbf{N}$ n c n en e fo ny ed cc cy nce e e ne depend on one e on y f e no con c e nd d fo of K x;x z n e x nd z fo p e dod e en ope o no nece y con o on e o n pe co p e on of e ope o ndeed f e e ope o e ep e en ed n e fo en e dependence of e e ne K x; on x po nd e n e of n c n en e n e nd d fo of O o 2 N

e pp en c y n co p n nece y o co p e e e e deco po on of f x z fo e e y x nd ppe o eq e O N[2] ope on e o of ec on cco p e n O N o

th