

On Generalized Gaussian Quadratures for Exponentials and Their Applications

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We introduce new families of Gaussian-type quadrature formulas for weighted integrals of exponential functions and consider their application to integration and interpolation of bandlimited functions.

We give a generalization of a representation theorem due to Caughy and to de Boer for the quadrature. For each positive measure, the quadrature is parameterized by eigenvalues of the Toeplitz matrix, constructed from the trigonometric moments of the measure. For a given accuracy

the form

$$c_k = \sum_{j=1}^M j e^{i j k}, \quad (1.1)$$

for $k = 1, 2, \dots, N$ and $M \geq N$, where $-1 < j < 1$ and $j > 0$. Cauchy's representation (1.1) has been the foundation for a number of algorithms for spectral estimation; in particular, [20] is known in electrical engineering literature as the Pisarenko method. In this paper we develop a fast algorithm for finding M , the phase T015

A method for constructing generalized Gaussian quadrature, or else limited to integral (with a fairly arbitrary measure) in solving exponential. Orthogonalizing eigenvalue and eigenvectors of a Toeplitz matrix, constructed from trigonometric moments of the measure and then computing the roots on the unit circle for appropriate eigenpolynomial. In particular, each eigenpolynomial with distinct roots give rise to an identity which, for small eigenvalue, provide with a Gaussian-type quadrature and also with a representation of positive definite Hermitian Toeplitz matrices. In the identities the size of the eigenvalue determine the accuracy of the quadrature formula.

It is noted that in the case of the eight leading to PSWF, the nodes of the corresponding Gaussian quadrature are zero (appropriately scaled to the interval $[-1, 1]$) of discrete PSWF corresponding to small eigenvalue.

As an application, we give the new quadrature 7(I)-4.9(n)-199.2gTj/Fx.7ng2(cited)cTj4o2a . 4.7

The paper is organized as follows. We present a brief description of the Piavko method to obtain the classical Carathéodory representation and we derive the estimate (1.2) in Section 2. In Section 3 we discuss generalized Gaussian quadrature for weighted integrals and prove some of their properties for weight supported in $[-1/2, 1/2]$. In Section 4 we introduce new families of Gaussian-type quadrature. We develop a fast algorithm in Section 5 to compute the nodes and weights of the quadrature. We solve the approximation problem (1.3)–(1.5) in Section 6 and we do it in the next section to obtain quadrature and interpolating bases for bandlimited functions. We also discuss a numerical example to illustrate the results. Finally, conclusions are presented in Section 9.

2. CARATHÉODORY REPRESENTATION

Carathéodory representation of the trigonometric moment problem and can be stated as follows (see [8, Chap. 4]).

THEOREM 2.1. *Given N complex numbers $\mathbf{c} = (c_1, c_2, \dots, c_N)$, not all zero, there exist unique $M \leq N$, positive numbers $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_M)$, and distinct real numbers $\theta_1, \theta_2, \dots, \theta_M$*

2.1. Algorithm 1: Method to Obtain M , θ , and ρ

(1) Given $c = (c_1, c_2, \dots, c_N)$, extend the definition of c_k to negative k as $c_{-k} = \overline{c_k}$ and define c_0 so that the $(N + 1) \times (N + 1)$ Toeplitz matrix T_N of elements $(T_N)_{kj} = c_{j-k}$, has nonnegative eigenvalues and at least one eigenvalue equal to zero.

(2) Define M as the rank of T_N . By construction, $M \leq N$. We also assume that M is the rank of the submatrix (2.1).

(3) Let T_M be the top left principal submatrix of order $M + 1$ of T_N . That is, the matrix T_M has elements $(c_{j-k})_{0 \leq k, j \leq M}$. Find the eigenvector q corresponding to the zero eigenvalue of T_M .

(4) Construct the polynomial (eigenpolynomial) whose coefficients are the entries of the eigenvector q . According to [8, p. 58], the M roots of this eigenpolynomial are distinct and have absolute value 1. The phase of the roots are the numbers θ_j .

(5) Find the eigenvector ρ by solving the Vandermonde system (2.1) for $k = 1, \dots, M$. The result, in addition, satisfies $\sum_k \rho_k = c_0$.

Remark 2.1. With the extension of the sequence c_k , (2.1) is valid for $|k| \leq N$. If $q = (q_0, \dots, q_M)$ is the eigenvector obtained in part (3) of Algorithm 2.1, then

$$\sum_{k=0}^M c_{k+s} q_k = 0, \tag{2.2}$$

for all $s, -N \leq s \leq 0$. In other words, we have found an order- M recurrence relation for the original sequence $\{c_k\}_{k=1}^N$.

Remark 2.2. In practice, we are interested in finding a stable order- M submatrix if M is small compared with N , or more generally, if most eigenvectors are smaller than the accuracy. However, in such cases, T_N has a large (numerical) null subspace that causes a numerical problem in determining c_0 , the rank M , and the eigenvector q .

Nevertheless, if the sequence c is the trigonometric moment of an appropriate eigenvector, we will be able to modify the procedure in order to obtain the phases θ_j in an efficient manner. In this setting, the phases and eigenvectors in a stable order- M submatrix can be thought of as the nodes and eigenvectors of a Gaussian-type quadrature for a weighted integral. Once the phases are obtained, Theorem 2.2 assures that the computation of the eigenvectors is a well-posed problem. In Section 5.2, we present a fast algorithm to obtain the eigenvectors by evaluating the characteristic polynomial at the nodes $e^{i\theta_j}$.

Remark 2.3. Given an Hermitian Toeplitz matrix T , let us consider its smallest eigenvalue $\lambda_{(N)}$

3. GENERALIZED GAUSSIAN QUADRATURES FOR EXPONENTIALS

3.1. Preliminaries: Chebyshev Systems

In this section we collect some definitions and results related to Chebyshev systems. We follow mostly Karlin and Studden [12] (see also [13]). Readers familiar with this topic may skip this section.

A family of $n + 1$ real-valued functions u_0, \dots, u_n defined on an interval $I = [a, b]$ is a *Chebyshev system* (T-system) if any nontrivial linear combination

THEOREM 3.5 [12, VI, Sec. 4]. *For the periodic T -system (3.5), a point*

In this section we tabling Cauchy's theorem and Theorem 3.7 from the previous section, to construct ν -different Gaussian quadrature formulae for integral ν -th ν -eight w . The quadrature are exact for trigonometric polynomial of appropriate degree.

We then generalize the type of quadrature formulae and develop a new family of Gaussian-type quadrature. This family of quadrature formulae is parameterized by the eigenvalue of the Toeplitz matrix.

$$T = \{t_{l-k}\}_{0 \leq k, l \leq N}. \tag{4.2}$$

Among the new quadrature formulae, only those corresponding to eigenvalue of small magnitude are of practical interest. In fact, the size of the eigenvalue determine the error of the quadrature formula. To compute the ν -eight and node of the quadrature, we develop a new algorithm which may be considered as a (major) modification of Algorithm 2.1. The new algorithm is described in Section 5. The main result of this section are gathered in Theorem 4.1.

We tabling Theorem 3.7 to write

$$t_k = \sum_{j=1}^N j e^{i \theta_j k} + c_0 (-1)^k, \quad \text{for } |k| \leq N, \tag{4.3}$$

for unique positive ν -eight θ_j and phase θ_j in $(-1, 1)$. Then, for an $A(z) = \sum_{|k| \leq N} a_k z^k$ in N , the space of Laurent polynomial of degree at most N , we have

$$\int_{-1}^1 A(e^{i \theta}) w(\theta) d\theta = \sum_{|k| \leq N} a_k t_k = \sum_{j=1}^N j A(e^{i \theta_j}) + c_0 A(-1), \tag{4.4}$$

for unique positive ν -eight θ_j and node $e^{i \theta_j}$.

Alternatively, using Cauchy's theorem (2.1) applied to the sequence $c_k = t_k$, $1 \leq k \leq N$,

$$\begin{aligned} \int_{-1}^1 A(e^{i \theta}) w(\theta) d\theta &= \sum_{j=1}^M j A(e^{i \theta_j}) + (t_0 - c_0) \frac{1}{2} \int_{-1}^1 A(e^{i \theta}) d\theta \\ &= \sum_{j=1}^M j A(e^{i \theta_j}) + c_0 \frac{1}{2} \int_{-1}^1 A(e^{i \theta}) d\theta, \end{aligned} \tag{4.5}$$

where $c_0 = \sum_{j=1}^M j$ and $\{e^{i \theta_j}\}$ are the roots of the eigenpolynomial corresponding to the smallest eigenvalue c_0 of T .

Note that (4.5) is again valid for all $A(z)$ in N and that the positive ν -eight θ_j and phase θ_j in $(-1, 1]$ are unique.

The ν -eight ν -different quadrature that may not coincide. However, by considering $w(\theta)$ proposed in $(-1/2, 1/2)$, (3.12) implies that w_0 in (4.4) decays exponentially fast with N and, since $\min w(\theta) = 0$ for $|\theta| \leq 1$, we have

$$\lim_{N \rightarrow \infty} c_0^{(N)} = 0, \tag{4.6}$$

4.2. Gaussian-Type Quadratures on the Unit Circle

In this section we present the main result of the paper. We define the Gaussian-type quadrature valid for an eigenvalue of the matrix T other than just the smallest eigenvalue λ_N . The quadrature allows us to select the desired accuracy and then, to construct accuracy-dependent families of quadratures.

The nodes of the quadrature in (4.5) are the roots of the eigenpolynomial corresponding to the least eigenvalue of T and, because of Cauchy's interlacing property, we know that the nodes are on the unit circle and that the weights are positive numbers. In our generalization, this standard property for the nodes and weights is no longer enforced. However, we will show that for nodes on the unit circle, the corresponding weights are real. Moreover, in all examples we have examined, for all small eigenvalues of T , the negative weights are associated with the nodes outside the support of the weights and are comparable in size with them. We believe this property to hold for a wide class of weights.

We prove the following

THEOREM 4.1. *Assume that the eigenpolynomial $V^{(s)}(z)$ corresponding to the eigenvalue $\lambda^{(s)}$ of T has distinct, nonzero roots $\{\lambda_j\}_{j=1}^N$. Then there exist numbers $\{w_j\}_{j=1}^N$ such that*

(i) *For all Laurent polynomials $P(z)$ of degree at most N ,*

$$\int_{-1}^1 P(e^{it})w(t) dt = \sum_{j=1}^N w_j P(\lambda_j) + \lambda^{(s)} \frac{1}{2} \int_{-1}^1 P(e^{it}) dt. \tag{4.12}$$

(ii) *For each root λ_k with $|\lambda_k| = 1$, the corresponding weight w_k is a real number and*

$$w_k = \int_{-1}^1 |L_k^s(e^{it})|^2 w(t) dt - \lambda^{(s)} \frac{1}{2} \int_{-1}^1 |L_k^s(e^{it})|^2 dt, \tag{4.13}$$

where

$$L_k^s(z) = \frac{V^{(s)}(z)}{(V^{(s)}(\lambda_k)(z - \lambda_k))} \tag{4.14}$$

is the Lagrange polynomial associated with the root λ_k .

(iii) *If $\lambda^{(s)}$ is a simple eigenvalue, then for $k = 1, \dots, N$, the weight w_k is nonzero and*

$$\frac{1}{w_k} = \sum_{\substack{l=1 \\ l \neq s}}^N \frac{V^{(l)}(\lambda_k) V^{(l)}(\lambda_k)}{(V^{(l)}(\lambda_k) - V^{(s)}(\lambda_k))}, \tag{4.15}$$

where $V^{(l)}(z) = \overline{V^{(l)}(z^{-1})}$ is the reciprocal polynomial of $V^{(l)}(z)$.

In particular, for each λ_k with $|\lambda_k| = 1$,

$$\frac{1}{w_k} = \sum_{\substack{l=1 \\ l \neq s}}^N \frac{|V^{(l)}(\lambda_k)|^2}{(V^{(l)}(\lambda_k) - V^{(s)}(\lambda_k))}. \tag{4.16}$$

(i) *If $\lambda^{(s)}$ is a simple eigenvalue and all roots λ_k are on the unit circle, then the set $\{w_k\}_{k=1}^N$ contains exactly s positive numbers and $N - s$ negative numbers.*

In particular, if $s = 0$ or $s = N$, then all w_k are negative or positive, respectively.

Remark 4.1. Our approach to obtain Gaussian quadrature does not use Stieltjes polynomials and is therefore substantially different than the one in [11]. We briefly explain the approach in [11]. Note that (4.9) and (4.10) show that the polynomials $\{V^{(k)}(z)\}$ are orthogonal with respect to both the usual inner product for trigonometric polynomials and the weighted inner product with weight $w(t)$. We can also construct Stieltjes polynomials $\{p_k(z)\}$ orthogonal with respect to $w(t)$ and check that each $p_k(z)$ has precisely k [26]. For an k , the roots of $p_k(z)$ are all in $|z| < 1$ [8].

Stieltjes polynomials and their reciprocal induce para-orthogonal polynomials [11],

$$p \quad e \quad r \quad a B_n B$$

For k outside the support of the measure, v_{ν} have been observed (Fig. 2, 3, and 5.8) that

$$\sum_{l: \binom{s}{l} > 0} |V^{(l)}(z_k)|^2$$

is a constant of moderate size.

Thus, the second term in (4.17) is $O(1/\binom{s}{l})$ and the v_{ν} might indeed be negative and orthogonal to the eigenspace.

Remark 4.5. For the v_{ν} with $|\nu| \leq 1$ in $(-1/2, 1/2)$ and 0 otherwise, the eigenpolynomial is the discrete PSWF. For the definition, v_{ν} are known that all eigenvalues are simple and that all eigenpolynomials are on the unit circle [23].

COROLLARY 4.1. *Under the assumptions of Theorem 4.1, it follows that the Toeplitz matrix T in (4.2) has the following representation as a sum of rank-1 Toeplitz matrices,*

$$(T - \binom{s}{l} I)_{kl} = \sum_{j=1}^N w_j z_j^{l-k},$$

where $\binom{s}{l}$, w_j , and z_j are as in (4.12).

This corollary should be compared with Remark 2.3 noting that, in the corollary, $\binom{s}{l}$ is not necessarily the least eigenvalue of T . For an alternative definition see [4].

Proof of Theorem 4.1. (1) For $x = (x_0, \dots, x_N) \in \mathbb{C}^{N+1}$, let us define

$$A_x(z) = \sum_{l=-L}^L x_{l+L} z^l, \quad \text{if } N = 2L$$

<

(3) Let $P \in \mathbb{C}[z]$; then $z^N P(z)$ is a polynomial of at most degree $2N$, and since $z^L V^{(s)}(z)$ is a polynomial of degree N , by Euclidean division, there exist polynomials $q(z)$ and $r(z)$ of degree at most N and $N - 1$ such that

$$z^N P(z) = z^L V^{(s)}(z)q(z) + r(z).$$

Then,

$$P(z) = V^{(s)}(z)Q(z) + R(z), \tag{4.19}$$

where $Q(z) \in \mathbb{C}[z]$ and $R(z)$ has the form $R(z) = \sum_{k=1}^N r_k z^{-k}$ and hence

$$\int_{-1}^1 R(e^{i t}) dt = 0.$$

Using the fact that $\{V^{(l)}\}_{l=0}^N$ is a basis of $L_{\mathbb{C}} e_{\mathbb{C}}$ it is

$$\overline{Q(e^{i t})} = \sum_{l=0}^N d_l V^{(l)}(e^{i t}),$$

where d_l are some complex coefficients.

Using (4.10) and (4.18), we multiply both sides of (4.19) by $w(t)$ and integrate to obtain

$$\int_{-1}^1 P(e^{i t})w(t) dt = \sum_{l=0}^N d_l \int_{-1}^1 V^{(l)}(e^{i t})w(t) dt$$

and then, considering $k = j$, (4.15) follows. Note that we need (s) to be simple to guarantee $V^{(l)} - V^{(s)} = 0, l = s$ in (4.20).

If we use the left hand side of (4.20) as the entries A_{kj} of a matrix A and let B be the matrix of entries

$$B_{lk} = V^{(l)}(x_k), \quad \text{where } 0 \leq l \leq N, l = s, \text{ and } 1 \leq k \leq N, \quad (4.21)$$

we can prove (4.20) by showing that $BA = B$ and that B is nonsingular.

For the latter claim, we simply check that the columns of B are linearly independent. Indeed, let $a_l, l = s$, be constants such that

$$\sum_{l=s} a_l V^{(l)}(x_k) = 0, \quad \text{for } k = 1, \dots, N.$$

It follows that the polynomial $P(z) = \sum_{l=s} a_l V^{(l)}(z)$ of degree L has the $N = 2L$ distinct roots x_k . Since P and $V^{(s)}$ have the same degree and the same N distinct roots, $P(z) = cV^{(s)}(z)$, for some constant c . By (4.9), $V^{(s)}(z)$ is orthogonal to all the other eigenpolynomials and so $a_l = 0$.

To show that $BA = B$, we substitute $P(z) = V^{(l)}(z)V^{(m)}(z)$ in (4.12) to obtain

$$\int_{-1}^1 V^{(l)}(e^{it}) \dots$$

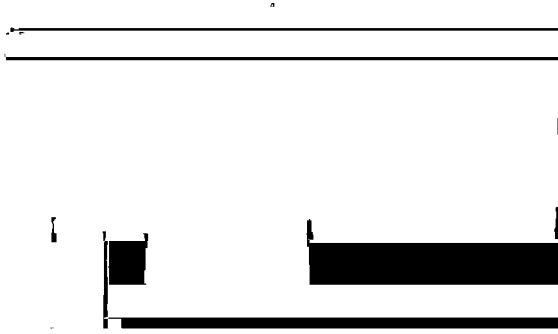


FIG. 2. Modified eigenpolynomial $e^{-i t(N/2)}V^{(30)}(e^{i t})$ on the interval $[-1, 1]$, where $N = 97$ and $V^{(30)}(e^{i t})$ is the eigenpolynomial corresponding to the eigenvalue $\lambda^{(30)}$ in Example 1. The phase factor $e^{-i tN/2}$ is introduced to make this function real.

EXAMPLE 1. Fix t_{ν} and consider the ν th eight

$$w(t) = \begin{cases} 1, & t \in [-a, a], \\ 0, & \text{elsewhere.} \end{cases} \quad (4.24)$$

For this ν th eight, the eigenpolynomial $V^{(l)}(e^{i t})$ of the $(N + 1) \times (N + 1)$ Toeplitz matrix T are the discrete PSWF [23]. Thus the eigenpolynomial $V^{(l)}(e^{i t})$ has all of its zeros on the unit circle. Moreover, it has exactly l zeros for t in the interval $(-a, a)$ and $N - l$ zeros for t in $[-1, 1]$. In this example, we have selected $N = 97$, $a = 1/6$, $c = 15$. We then construct the matrix T and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(30)} = 9.77306136381891632828 \cdot 10^{-16}. \quad (4.25)$$

The eigenpolynomial $V^{(30)}(e^{i t})$ is shown in Fig. 2 and 3. Location of the zeros on the unit circle are displayed in Fig. 4. We then determine the quadrature formula corresponding to this eigenvalue and tabulate the ν th eight in Table I. Note that the ν th eight is defined on the interval $[-1/6, 1/6]$

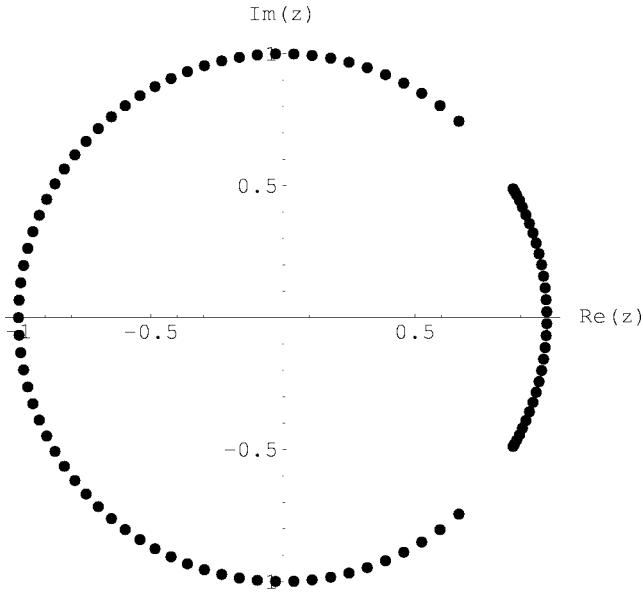


FIG. 4. Location of the e o on the unit circle for the eigenpolynomial $V^{(30)}$ in Example 1.

TABLE I
Table of Weights for the Quadrature Formula with $\lambda^{(30)}$ in Example 1

#	Weight	#	Weight
1	$-1.0328 \cdot 10^{-17}$	50	0.04437549133235668283
2	$-1.0328 \cdot 10^{-17}$	51	0.04419611220330997984
3	$-1.0329 \cdot 10^{-17}$	52	0.04382960375644760677
\vdots	\vdots	53	0.04325984471286061543
33	$-1.3518 \cdot 10^{-17}$	54	0.04246105337417774134
34	$-1.6030 \cdot 10^{-17}$	55	0.04139574827622469674
35	0.00580295532842819966	56	0.04001188663952018400
36	0.01310603337477264417	57	0.03823923547752508920
37	0.01959211245475268191	58	0.03598544514201341779
38	0.02506789313597245367	59	0.03313334531810570720
39	0.02954323947353217723	60	0.02954323947353217723
40	0.03313334531810570720	61	0.02506789313597245367
41	0.03598544514201341779	62	0.01959211245475268191
42	0.03823923547752508920	63	0.01310603337477264417
43	0.04001188663952018400	64	0.00580295532842819966
44	0.04139574827622469674	65	$-1.6030 \cdot 10^{-17}$
45	0.04246105337417774134	66	$-1.3518 \cdot 10^{-17}$
46	0.04325984471286061543		
47	0.04382960375644760677		
48	0.04419611220330997984		
49	0.04437549133235668283		

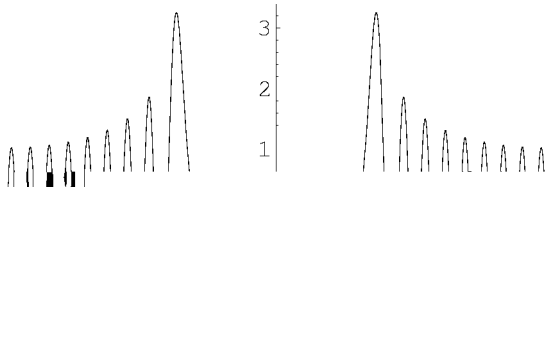


FIG. 5. Modified eigenpolynomial (see Fig. 2) on the interval $[-1, 1]$ corresponding to the eigenvalue $\lambda^{(28)}$ in Example 2.

EXAMPLE 2. We consider the weight

$$w(t) = \begin{cases} |t|/a, & t \in [-a, a], \quad a = 1/2, \\ 0, & \text{elsewhere.} \end{cases} \quad (4.26)$$

In this example we have selected $N = 61$, $a = 1/4$, $c = 15$. We then construct the matrix T and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(28)} = 1.11598931688523706280 \cdot 10^{-14}. \quad (4.27)$$

The eigenpolynomial $V^{(28)}(e^{it})$ is shown in Fig. 5 and 6.

EXAMPLE 3. We consider a non-symmetric weight

$$w(t) = \begin{cases} 1 + t/a, & t \in [-a, a], \quad a = 1/2, \\ 0, & \text{elsewhere.} \end{cases} \quad (4.28)$$

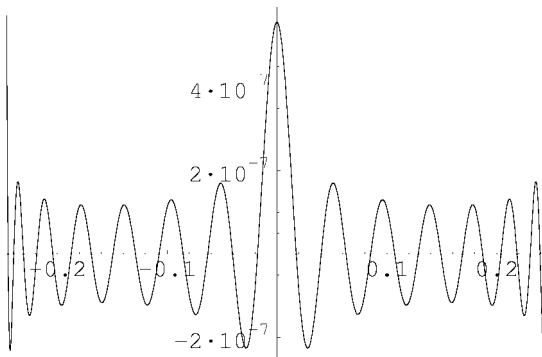


FIG. 6. The same function of Fig. 5 on the interval $[-1/4, 1/4]$.

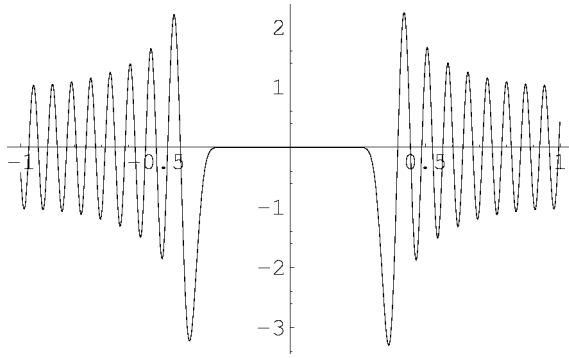


FIG. 7. Modified eigenpolynomial (see Fig. 2) on the interval $[-1, 1]$ corresponding to the eigenvalue $\lambda^{(28)}$ in Example 3.

In this example we have selected $N = 61$, $a = 1/4$, $c = 15$. We then construct the matrix T and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(28)} = 4.68165338379692121389 \cdot 10^{-15}. \tag{4.29}$$

The eigenpolynomial $V^{(28)}(e^{it})$ is shown in Fig. 7 and 8. Although we do not have a proof at the moment, it appears that the eigenvalues of eight of the highest eigenpolynomial corresponding to small eigenvalues mimic the behavior of the discrete PSWF with respect to location of zeros. In Example 3 we know that all zeros are on the unit circle due to Theorem 4.2 and 4.3.

In Table II we illustrate the performance of quadrature for different bandlimit c . This table should be compared with [29, Table 1]. The performance of both sets of quadrature is very similar. Yet the quadrature are quite different and can be better compared Table III with [29, Table 5]. Although the accuracy is almost identical, approximately 10^{-10}

TABLE II
Quadrature Performance for Varying Bandlimits

c	# of node	Max. im. m. e. o.
20	13	$1.2 \cdot 10^{-7}$
50	24	$1.1 \cdot 10^{-7}$
100	41	$1.6 \cdot 10^{-7}$
200	74	$1.8 \cdot 10^{-7}$
500	171	$1.4 \cdot 10^{-7}$
1000	331	$2.4 \cdot 10^{-7}$
2000	651	$1.2 \cdot 10^{-7}$
4000	1288	$3.7 \cdot 10^{-7}$

5. A NEW ALGORITHM FOR CARATHÉODORY REPRESENTATION

5.1. Algorithm 2

We now describe an algorithm for computing quadrature nodes via a Carathéodory-type approach based on Theorem 4.1. It is easy to see that, although the elements are similar to those with

TABLE III
Quadrature Nodes for Exponentials with Maximum Bandlimit $c = 50$

Node	Weight
-0.99041609489889	2.42209284787E-02
-0.95238829377394	5.04152570050E-02
-0.89243677566550	6.82109308489E-02
-0.81807124037876	7.96841731718E-02
-0.73438712699465	8.71710040243E-02
-0.64454148960251	9.22000859355E-02
-0.55050369342444	9.56668891250E-02
-0.45355265507507	9.80920675810E-02
-0.35456254990620	9.97843340729E-02
-0.25416536256280	1.00930070892E-01
-0.15284664158549	1.01641529848E-01
-0.05100535080412	1.01982696564E-01
0.05100535080412	1.01982696564E-01
0.15284664158549	1.01641529848E-01
0.25416536256280	1.00930070892E-01
0.35456254990620	9.97843340729E-02
0.45355265507507	9.80920675810E-02
0.55050369342444	9.56668891250E-02
0.64454148960251	9.22000859355E-02
0.73438712699465	8.71710040243E-02
0.81807124037876	7.96841731718E-02
0.89243677566550	6.82109308489E-02
0.95238829377394	5.04152570050E-02
0.99041609489889	2.42209284787E-02

Pi anko' method, the corresponding algorithm are substantially different. We plan to add the implication for signal processing in a separate paper.

(1) Given t_k , the trigonometric moment of a measure $\mu_{\mathbb{T}^N}$ construct the $(N+1) \times (N+1)$ Toeplitz matrix T_{N+1} with element $(T_{N+1})_{kj} = t_{j-k}$. This matrix is positive definite and has a large number of small eigenvalues.

(2) For a given accuracy ϵ , compute the inverse of the Toeplitz matrix $T_N - I$. For a self-adjoint Toeplitz matrix, it is sufficient to solve $(T_N - I)p = \text{prj } V \text{ } Q \text{ } p \text{ } \mathbb{T}^N$.

If φ_k e de ne

$$Q(z) = \prod_{k=1}^M (z - \alpha_k) = \sum_{k=0}^M q_k z^k, \tag{5.2}$$

then, fo an pol nomial P of deg ee at mo t $M - 1$,

$$\frac{P(z)}{Q(z)} = \sum_{r=1}^M \frac{P(\alpha_r)}{Q'(\alpha_r)(z - \alpha_r)}.$$

Th , fo $|z| < \min |\alpha_r|^{-1}$,

$$\frac{z^{M-1} P(z^{-1})}{z^M Q(z^{-1})} = \sum_{r=1}^M \frac{P(\alpha_r)}{Q'(\alpha_r)} \sum_{k=0}^{+\infty} \alpha_r^k z^k = \sum_{k=0}^{+\infty} \left(\sum_{r=1}^M \frac{P(\alpha_r)}{Q'(\alpha_r)} \alpha_r^k \right) z^k. \tag{5.3}$$

No φ_k choo e P to be the . niq e pol nomial φ_k ith $P(\alpha_r) = \alpha_r Q'$

This algorithm is equivalent to the following factorization of the inverse of the Vandermonde matrix in terms of a diagonal matrix, its transpose V^t , and a triangular Hankel matrix,

$$V^{-1} = \begin{matrix} \frac{1}{Q(r_1)} & \cdots & 0 & & q_1 & q_2 & \cdots & q_M \\ & & \ddots & & q_2 & \cdots & q_M & 0 \\ & & & & \vdots & & & \vdots \\ 0 & \cdots & \frac{1}{Q(r_M)} & & q_M & \cdots & 0 & 0 \end{matrix} V^t \quad (5.5)$$

This description is a particular case of the inversion formulae for $L_{\mathbb{F}_q}$ -Vandermonde [21] or closely to Vandermonde matrices [9, Corollary 2.1, p. 157]. We can state the result as (see [21, p. 548])

$$V^{-1} = \begin{matrix} & & & & -y_2 & -y_3 & \cdots & 1 \\ & x_1 & \cdots & 0 & -y_3 & \cdots & 1 & 0 \\ & & & \ddots & \vdots & & & \vdots \\ & & & & & & & 0 \\ 0 & \cdots & x_M & & & & \cdots & 0 \\ & & & & 1 & \cdots & 0 & 0 \end{matrix} V^t,$$

where the vectors $\mathbf{x} = (x_1, \dots, x_M)^t$ and $\mathbf{y} = (y_1, \dots, y_M)^t$ are a solution of

$$V\mathbf{x} = (0, \dots, 1)^t \quad \text{and} \quad V^t\mathbf{y} = [r^M]_{r=1}^M.$$

Since r are the roots of $Q(z)$, we can take $\mathbf{y} = -(q_0, \dots, q_{M-1})^t$, and if $B(z) = z^M$ in (5.4), then $P(z) = 1$ and $\mathbf{x} = (1/Q(r_1), \dots, 1/Q(r_M))^t$.

Remark 5.1. For Algorithm 5.1, we first obtained the eigenvectors corresponding to an eigenvalue. Then, step (1) of the Vandermonde algorithm is already accomplished and step (2) can be performed using the FFT. For the moment, the nodes r_k belong to the unit circle and, via the nearest spaced fast Fourier transform, we have a fast algorithm to obtain the eigenvectors.

Remark 5.2. As an example, we use this approach to describe the solution of the Vandermonde system with nodes $r = e^{i2\pi(r-1)/M}$, $1 \leq r \leq M$. In this case, $Q(z) = 1 - z^M$

Proof of Theorem 6.1. Let

$$u(y) = \int_{-1}^1 (t)e^{i ty} dt,$$

and, for each m , define the spline of order $2m - 1$ interpolating $u(y)$ at the integers,

$$a(y) = \sum_k u(k)L_{2m-1}(y - k) = \int_{-1}^1 (t)S_{2m-1}(y, e^{it}) dt.$$

By (6.7),

$$|u(y) - a(y)| \leq 3 \int_{-1}^1 (t)|t|^{2m} dt \leq 3^{-2m} \leq \frac{1}{4},$$

where $e^{-1} = \int_{-1}^1 (t) dt$. We choose m such that $3^{-2m} \leq \frac{1}{4}$.

On the other hand, for each N , Theorem 3.7 allows us to represent the moment $u(k)$, $|k| \leq N$,

$$u(k) = \int_{-1}^1 (t)e^{ikt} dt = \sum_{j=1}^N w_j e^{i jk} + w_0 (-1)^k, \tag{6.9}$$

where

$$w_0 = \frac{4^{-1}}{2 + (2 + \frac{1}{3})^N + (2 - \frac{1}{3})^N}. \tag{6.10}$$

Let

$$\tilde{u}(y) = \sum_{j=1}^N w_j e^{i jy};$$

then $u(k) = \tilde{u}(k) + w_0 (-1)^k$ for $|k| \leq N$, and defining

$$\tilde{a}(y) = \sum_k \tilde{u}(k)L_{2m-1}(y - k) = \sum_{j=1}^N w_j S_{2m-1}(y, e^{ij}),$$

(6.7) gives the estimate

$$|\tilde{u}(y) - \tilde{a}(y)| \leq 3 \sum_{j=1}^N w_j |y - j|^{2m} \leq 3^{-2m} (u(0) - w_0) \leq 3^{-2m} \leq \frac{1}{4}.$$

We have hence that $u(y)$ is close to $a(y)$ and $\tilde{u}(y)$ is close to $\tilde{a}(y)$. To finish the proof, we need to show that $|a(y) - \tilde{a}(y)| < \frac{1}{2}$, for $|y| \leq dN + 1$. Now,

$$\begin{aligned} a(y) - \tilde{a}(y) &= \sum_{|k| \leq N} w_0 (-1)^k L_{2m-1}(y - k) + \sum_{|k| > N} (u(k) - \tilde{u}(k)) L_{2m-1}(y - k) \\ &= w_0 S_{2m-1}(y, e^{i \cdot}) + \sum_{|k| > N} (u(k) - \tilde{u}(k) - w_0 (-1)^k) L_{2m-1}(y - k) \end{aligned}$$

and

$$\begin{aligned} |u(k) - \tilde{u}(k) - w_0 (-1)^k| &\leq |u(k)| + |\tilde{u}(k)| + w_0 \leq \sum_{j=0}^N w_j + \sum_{j=1}^N w_j + w_0 \\ 2u(0) &= 2 \leq 1, \end{aligned}$$

where we used (6.9).

Since J_{2n} is an even function, we have

$$v(x) = \int_{-1}^1 \bar{w}(t) J_{2n}(cx) dt. \quad (7.4)$$

Using

$$J_{2n}(t) = \frac{(-1)^n}{t^{2n}}$$

where

$$\tilde{v}_j = \sum_{k=1}^M w_k j(t_k), \tag{7.13}$$

and the nodes t_k and the weights w_k are the same as in (1.4).

For large c , the spectrum of F_c can be divided into two groups. The first group contains approximately $2c/\alpha$ eigenvalues with absolute value close to 1. The second group contains $\log c$ eigenvalues whose absolute values make an exponential transition from 1 to 0. The third group consists of exponentially decaying eigenvalues that are close to zero. For precise statements see [14, 24, 25, 29].

Therefore, it follows from (7.12) that, for the $2c/\alpha$ eigenfunction, the integral in (7.11) is well approximated by the quadrature in (7.13). To prove (7.12), see (7.10), to wit

$$v_j - \tilde{v}_j = \frac{1}{j} \int_{-1}^1 \int_{-1}^1 w(t) e^{ic t d} - \sum_{k=1}^M w_k e^{ic k t} j(t) dt. \tag{7.14}$$

Since $|t| \leq 1$, we have

$$\left| \int_{-1}^1 w(t) e^{ic t d} - \sum_{k=1}^M w_k e^{ic k t} \right|, \tag{7.15}$$

and $j_2 = \text{arXiv:1404.0052 [math]} \text{TD}(0.0052 \text{ Tc}(1.41 \text{e}012) \text{Tj}67 \text{ } 8 \text{Tj}/(i)0.9(\text{m Tm})8.6483 \text{ } 0 \text{ TD-94h5)T1 T}$

In considering bandlimited function $f(x)$ in the PSWF (see [15, 24], and a more recent paper by [16]) in (7.9) the eigenvalues

$$j^2 =$$

By setting

$$I = w_l \sum_{j=0}^{M-1} j(b/c) j(t_l), \tag{8.18}$$

and observing that $|M|$ and that $|j| \leq |M|$ for $j > M$, we obtain (8.5) and (8.6). ■

We now construct a linear combination of the function $\{e^{ict_l x}\}_{l=1}^M$. First, let us consider the following algebraic eigenvalue problem,

$$\sum_{l=1}^M w_l e^{ict_l t_l} j(t_l) = j j(t_m), \tag{8.19}$$

where t_l and w_l are the same as in (8.1). By solving (8.19), we find j and $j(t_l)$. We then consider function $j, j = 1, \dots, M$, defined for an x as

$$j(x) = \frac{1}{j} \sum_{l=1}^M w_l e^{icx t_l} j(t_l). \tag{8.20}$$

The function j in (8.20) are linear combination of the exponential functions

Let $\{e^{icx t_l}\}_{l=1}^n$ be a set of n exponential functions. We define the function $R_k, k = 1, \dots, M$, as

$$R_k(x) = \sum_{l=1}^M r_{kl} e^{icx t_l}, \tag{8.27}$$

where

$$r_{kl} = \sum_{j=1}^M w_k \frac{1}{j} (t_k)^{j-1} - \sum_{j=1}^M \overline{w_k} q_k^j \frac{1}{j} q_l^{j-1} \overline{w_l}. \tag{8.28}$$

By direct evaluation in (8.19) and (8.23), we see that the function R_k is interpolating,

$$R_k(t_m) = \delta_{km}. \tag{8.29}$$

Let us show that the integral of $R_k(t)e^{iat}$, where $|a| < c$, yields a one-point quadrature rule of accuracy $O(\epsilon)$.

PROPOSITION 8.3. For $|a| < c$, let

$$I_k = \int_{-1}^1 R_k(t) e^{iat} dt - w_k e^{iat_k}. \tag{8.30}$$

Then we have

$$|I_k| \leq \frac{2}{M} \frac{\max_{k=1, \dots, M} |w_k|}{\min_{k=1, \dots, M} |w_k|} \epsilon, \tag{8.31}$$

where $\epsilon = \sqrt{\frac{2}{M}}$.

PROPOSITION 8.4. For every b , $|b| < c$, let us consider the function

$$b(t) = e^{ibt} - \sum_{k=1}^M e^{ibt_k} R_k(t). \tag{8.34}$$

Then, for every $|a| < c$, we have

$$\left| \int_{-1}^1 b(t)e^{iat} dt \right| \leq 1 + M \frac{\max_{k=1, \dots, M} |w_k|}{\min_{k=1, \dots, M} |t_k|}. \tag{8.35}$$

Proof. Using (8.30), we have

$$\int_{-1}^1 b(t)e^{iat} dt = \int_{-1}^1 e^{i(b+a)t} dt - \sum_{k=1}^M w_k e^{i(b+a)t_k} - \sum_{k=1}^M e^{ibt_k} R_k(t), \tag{8.36}$$

where

$$R_k = \int_{-1}^1 R_k(t)e^{iat} dt - w_k e^{iat_k}. \tag{8.37}$$

Applying (8.1), we obtain

$$\left| \int_{-1}^1 b(t)e^{iat} dt \right| \leq 2 + \overline{M}. \tag{8.38}$$

The estimate (8.35) then follows from Proposition 8.3. ■

Remark 8.2. Using the function R_k , $k = 1, \dots, M$, on a hierarchy of intervals, it is possible to construct a multiresolution basis (for a finite number of scales) similar to multiresolution wavelets. We will consider such construction and its application elsewhere.

8.1. Examples

For the eight

$$R_k(t) = \begin{cases} 1, & t \in [-a, a], \quad a = 1/2, \\ 0, & \text{otherwise,} \end{cases} \tag{8.39}$$

we construct a 30-node quadrature formula so that (8.1) is satisfied with $\epsilon = 10^{-15}$. We

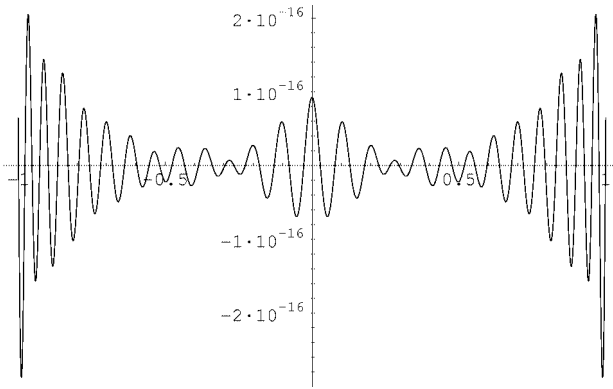


FIG. 9. E_0 in (8.1) for Example 1.

φ_9 is the Legendre polynomial of degree 9. The function φ_9 is not periodic and $\varphi_9 \in C^\infty$.

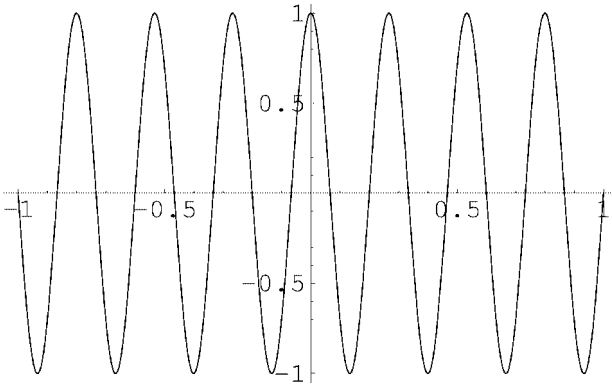


FIG. 11. Function $g_1(t)$ on the interval $[-1, 1]$.

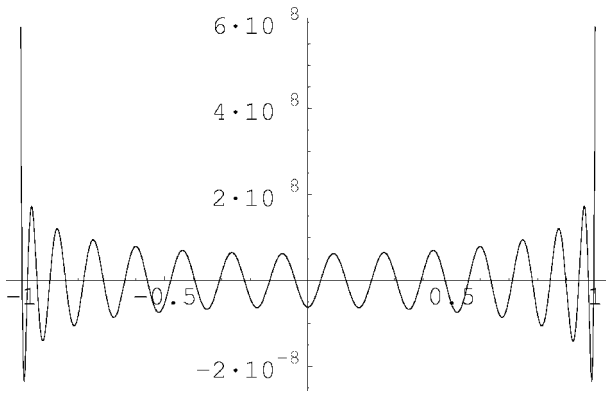


FIG. 12. Difference $g_1(t) - \bar{g}_1(t)$ on the interval $[-1, 1]$.

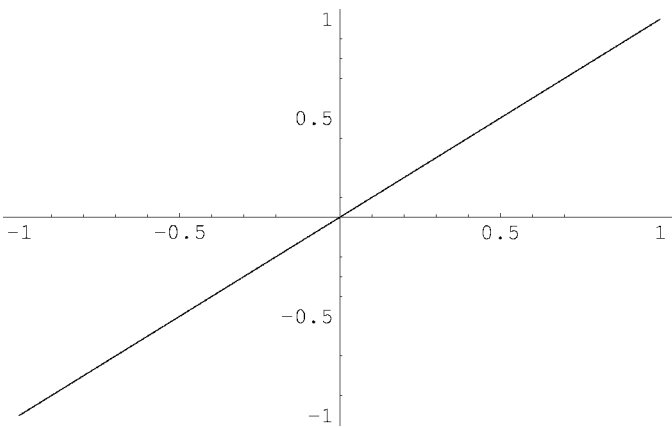


FIG. 13. Function $g_2(t)$ on the interval $[-1, 1]$.

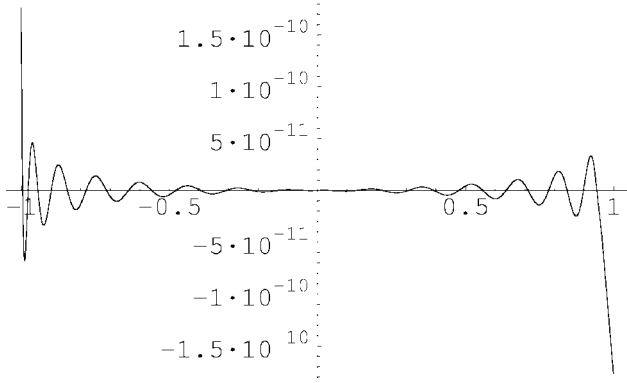


FIG. 14. Difference $g_2(t) - \tilde{g}_2(t)$ on the interval $[-1, 1]$.

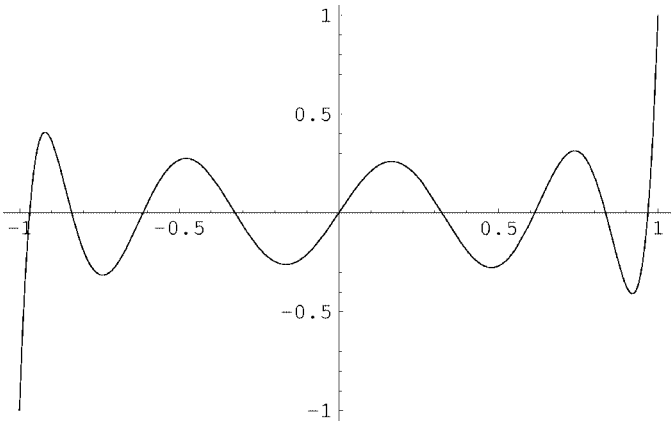


FIG. 15. Function $g_3(t)$ on the interval $[-1, 1]$.

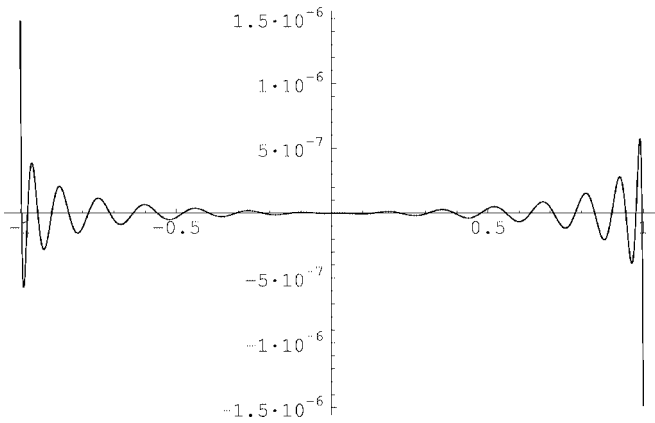


FIG. 16. Difference $g_3(t) - \tilde{g}_3(t)$ on the interval $[-1, 1]$.

exponential decay (see Fig. 1). For small eigen values, the quadrature is of practical interest.

The remarkable feature of the quadrature is that the half node outside the support of the measure and, a fortiori, the corresponding weights are negative and small,

Proof of Theorem 2.2. We take (2.1) to extend the definition of c_k and $c_{-k} = \overline{c_k}$ for $k = 1, \dots, N$ and $c_0 = \sum_{j=1}^M q_j$. We then define the Toeplitz matrix \mathbf{T}_N , $(\mathbf{T}_N)_{kj} = (c_{j-k})_{0 \leq k, j \leq N}$, and the polynomial

$$Q(z) = \prod_{j=1}^M (z - e^{i\theta_j}) = \sum_{k=0}^M q_k z^k.$$

Then $\mathbf{q} = (q_0, \dots, q_M, 0, \dots, 0)^t$

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