APPM 3570/STAT 3100 Exam 3 Fall 2022

NAME: ____

SECTION: 001 at 9:05 am

Instructions:

1. Notes, your text and other books, cell phones, and other electronic devices are not permitted, except as needed to view and upload your work, and except calculators.
2. Calculators are permitted.
3. Justify your answers, show all work.
4.

(c) Suppose the number of foreign fragments in a portion of peanut butter is a random variable with a mean of 3 and a variance of 3. Suppose a random sample of 50 portions of peanut butter are collected and the average number of foreign fragments in a portion is calculated. What is the probability of observing an average of 3.6 or more fragments?

Solution:

(a) (10 points)

(i) (5 points) The possible outcomes of X_1 and X_3 are:

Outcome	X_1	X_3	Prob.
YYG	1	0	2/10 * 1/9 * 8/8 = 16/720
YGY	1	1	2/10 * 8/9 * 1/8 = 16/720
YGG	1	0	2/10 * 8/9 * 7/8 = 112/720
GYY	0	1	8/10 * 2/9 * 1/8 = 16/720
GYG	0	0	8/10 * 2/9 * 7/8 = 112/720
GGY	0	1	8/10 * 7/9 * 2/8 = 112/720
GGG	0	0	8/10 * 7/9 * 6/8 = 336/720

The joint pmf of X_1 and X_3 is:

		X_3	
		0	1
X_1	0	448/720	128/720
	1	128/720	16/720

The joint pmf of X_1 and X_3 after simplification is:

		X_3	
		0	1
X_1	0	28/45	8/45
	1	8/45	1/45

$P(X_3 = 1 | X_1 = 1) = \frac{P(X_3=1, X_1=1)}{P(X_3=1)} = \frac{\frac{1}{45}}{\frac{9}{45}} = \frac{1}{9}.$

(ii) (5 points) $P(X_1 = X_3) = P(X_1 = 0, X_3 = 0) + P(X_1 = 1, X_3 = 1) = \frac{28}{45} + \frac{1}{45} = \frac{29}{45}.$

(b) (12 points)

(i) (4 points) We are given that $X_i^{(i)1}$ $N(434, {}^2 = 49)$, for $i = 1, 2, \ldots$ Therefore,

(c) (6 points) By the Central Limit Theorem, $\overline{X} \overset{approx}{\sim} N(3, \sigma^2 = \frac{3}{50})$.
$P(\overline{X} > 3.6) = P(Z > \frac{3.6-3}{.245}) \approx P(Z > 2.45) = 1 - \Phi(2.45) \approx 1 - .9929 = .0071.$

Problem 2. (28 points) Let (X, Y) be jointly distributed random variables with conditional pdf given by:

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{x^2} & 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

and marginal pdf of X given by:

$$f_X(x) = \begin{cases} 5x^4 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the joint pdf of X and Y.

(b) Find the marginal pdf of $Y.$

(c) Find $E[Y|X]$ and use it to find the expectation of $Y.$

(d) Find $\text{Cov}(X, Y)$.

Solution:

(a) (7 points)

$$f_{X,Y}(x, y) = f_{Y|X}(y|x) f_X(x)$$

$$f_{X,Y}(x, y) = \begin{cases} 10x^2 y & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) (7 points)

$$f_Y(y) = \int_y^1 10x^2 y \, dx$$
$$= \left. \frac{10x^3 y}{3} \right|_y^1$$
$$= \frac{10y}{3} - \frac{10y^4}{3}$$

$$f_Y(y) = \begin{cases} \frac{10y}{3} - \frac{10y^4}{3} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3. (18 points) If X and

(b) Suppose that the time it takes (in hours) to replace a failed battery is uniformly distributed over (0, .8), independently. Approximate the probability that all batteries have failed before 1000 hours have passed.

Solution:

(a) (11 points)
Let X_i be the lifetime of the ith battery, $i = 1, 2, \ldots, 200.$
$E[X_i] = 4,\ Var(X_i) = 16$
By the CLT, $\sum_{n=1}^{200} X_i \overset{approx}{\sim} N(800, 3200)$

$$P\left(\sum_{n=1}^{200} X_i > 810\right) \approx P\left(Z > \frac{810 - 800}{\sqrt{3200}}\right)$$
$$= P(Z > .1768)$$
$$\approx 1 - \Phi(.18)$$
$$= 1 - .5714 = .4286$$

(b) (15 points)
Let R_i be the time needed to replace the ith battery, $i = 1, 2, \ldots, 200.$
$E[R_i] = .4,\ Var(R_i) = .8^2 \times 1/12 = .0533$

$E[\sum_{n=1}^{199} R_i] = 79.6,\ Var(\sum_{n=1}^{199} R_i) = 10.6067$

By the CLT, $\sum_{n=1}^{200} X_i + \sum_{n=1}^{199} R_i \overset{approx}{\sim} N(879.6, 3210.61)$

$$P\left(\sum_{n=1}^{200} X_i + \sum_{n=1}^{199} R_i < 1000\right) \approx P\left(Z < \frac{1000 - 879.6}{\sqrt{3210.61}}\right)$$
$$= P(Z < 2.1249)$$
$$\approx \Phi(2.12)$$
$$= .983$$

Bonus Problem. (3 points) Let X be a random variable with mean 50 and variance 25. If X is the number of cars produced in a week at a particular auto manufacturing plant, find a lower bound on the probability that the production of cars in a week is between 30 and 70.

Solution: $P(|X - 50| \geq 20) \leq \frac{Var(X)}{400} = \frac{25}{400} = \frac{1}{16}$.
$P(|X - 50| < 20) \geq 1 - \frac{1}{16} = \frac{15}{16}$.