1. [2360/041923 (20 pts)] Consider the initial value problem $y''' - 2y'' = 64e^{-2t}$; $y(0) = y'(0) = 0$; $y''(0) = 4$.

   (a) (12 pts) Solve the initial value problem, using the methods of Chapter 4 (that is, do not use Laplace transforms).

   (b) (8 pts) Write the initial value problem as a system of first order differential equations. If possible, write the system, including the initial conditions, in the form $\vec{x}\,' = A\vec{x} + \vec{f}$; $\vec{x}(0) = \vec{x}_0$. If not possible, say so.

SOLUTION:

(a) The characteristic equation is $r^3 - 2r^2 = r^2(r-2) = 0 \implies r = 0$ (multiplicity 2); $r = 2$ (multiplicity 1). Basis for the solution space of the homogeneous equation is $\{1, t, e^{2t}\}$. Let $y_p = Ae^{-2t}$ and substitute into the nonhomogeneous equation to get

$$-8Ae^{-2t} - 2\cdot 4Ae^{-2t} = -16Ae^{-2t} = 64e^{-2t} \implies A = -4$$

The general solution is $y(t) = c_1 + c_2 t + c_3 e^{2t} - 4e^{-2t}$. Applying the initial conditions yields

$$y(0) = c_1 + c_3 - 4 = 0$$

$$y'(t) = c_2 + 2c_3 e^{2t} + 8e^{-2t} \implies y'(0) = c_2 + 2c_3 + 8 = 0$$

$$y''(t) = 4c_3 e^{2t} - 16e^{-2t} \implies y''(0) = 4c_3 - 16 = 4$$

$$4c_3 = 20 \implies c_3 = 5$$

$$c_2 = -8 - 2(5) = -18$$

$$c_1 = 4 - 5 = -1$$

The solution to the initial value problem is $y(t) = 5e^{2t} - 4e^{-2t} - 18t - 1$.

(b) Let $u_1 = y$; $u_2 = y'$; $u_3 = y''$. Then

$$u_1' = y' = u_2$$

$$u_2' = y'' = u_3$$

$$u_3' = y''' = 2y'' + 64e^{-2t} = 2u_3 + 64e^{-2t}$$

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 64e^{-2t} \end{bmatrix}; \quad \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

which is in the form $\vec{x}\,' = A\vec{x} + \vec{f}$; $\vec{x}(0) = \vec{x}_0$.

2. [2360/041923 (24 pts)] On a separate page in your bluebook, write the letters (a) through (l) in a column. Then for the following questions, write the word TRUE or FALSE next to each letter, as appropriate. No partial credit given and no work need be shown. If you do any work to come up with your answers, please do it elsewhere - do not include it in your list of answers (this helps with grading).

   An harmonic oscillator consisting of a 2-kg mass attached to a spring is horizontally aligned on a table with x measuring the displacement of the mass from its equilibrium position. The damping force is given as $-2p\dot{x}$, where p is a nonnegative real number, and the circular frequency of the oscillator is $\omega_0 = \sqrt{3}$.

   (a) If the oscillator is unforced, the differential equation governing the motion is $2\ddot{x} + 2p\dot{x} + \sqrt{3}x = 0$.

   (b) If $0 \le p < 2\sqrt{3}$, the mass will pass through its equilibrium position more than once if it is given a nonzero initial velocity.

   (c) If $p = 0$ and the oscillator is forced by4is forced by4is 5 Td [(y)]TJ/F13 6.9738 Tf 5.242 4.114 Td [(00)]TJ/F8 9.su 9.su 9.ra

(i) If the oscillator is forced by $f(t) = \frac{t}{t+1}$

(b) The form of the particular solution is $y_p = At^2 + Bt + C$. Substituting this into the DE yields

$$2(2A) - 12(2At + B) + 18(At^2 + Bt + C) = 18At^2 + (-24A + 18B)t + 4A - 12B + 18C = 9t^2 - 15$$

$$18A = 9 \Rightarrow A = \frac{1}{2}$$

$$-24A + 18B = 0 \Rightarrow -24\left(\frac{1}{2}\right) = -18B \Rightarrow B = \frac{2}{3}$$

$$4A - 12B + 18C = -15 \Rightarrow 4\left(\frac{1}{2}\right) - 12\left(\frac{2}{3}\right) + 15 = -18C \Rightarrow C = -\frac{1}{2}$$

Thus $y_{p_1} = \frac{1}{2}t^2 + \frac{2}{3}t - \frac{1}{2}$ is a particular solution.

(c) Before proceeding, we put the differential equation into the form $y'' - 6y' + 9y = 6t^{-1}e^{6t}$ and let $y_1 = e^{3t}$; $y_2 = te^{3t}$. The right hand side is $g(t) = 6t^{-1}e^{3t}$ and the particular solution will have the form $y_{p_2} = v_1y_1 + v_2y$