<u>**SOLUTION:**</u>

(a) There are three vectors in $\mathbb{R}^3$, a vector space of dimension 3. Check for linear independence:

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}-2\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \iff \begin{bmatrix}1 & 0 & -2\\1 & 1 & -1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{vmatrix}1 & 0 & -2\\1 & 1 & -1\\0 & 1 & 1\end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix}1 & -1\\1 & 1\end{vmatrix} + 1(-1)^{2+1}\begin{vmatrix}0 & -2\\1 & 1\end{vmatrix} = -2+2 = 0$$

The vectors are linearly dependent and thus cannot form a basis for $\mathbb{R}^3$.

(b) i. W is not a subspace. W is closed under vector addition (the sum of two integers is an integer) but is not closed under scalar multiplication. For example,

$$\vec{u} = I \in W \quad \text{but} \quad \sqrt{2}\,\vec{u} = \begin{bmatrix}\sqrt{2} & 0 & 0\\0 & \sqrt{2} & 0\\0 & 0 & \sqrt{2}\end{bmatrix} \notin W$$

ii. W is a subspace. Use linear combinations to check for closure. Let $\alpha, \beta \in \mathbb{R}$ and

$$\vec{u} = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} \in W \implies 4u_1 - 3u_2 + 9u_3 = 0 \quad \text{and} \quad \vec{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} \in W \implies 4v_1 - 3v_2 + 9v_3 = 0$$

Then $\alpha\vec{u} + \beta\vec{v} = \begin{bmatrix}\alpha u_1 + \beta v_1\\ \alpha u_2 + \beta v_2\\ \alpha u_3 + \beta v_3\end{bmatrix}$ and

$$\begin{aligned}4(\alpha u_1 + \beta v_1) - 3(\alpha u_2 + \beta v_2) + 9(\alpha u_3 + \beta u_3) &= \alpha(4u_1 - 3u_2 + 9u_3) + \beta(4v_1 - 3v_2 + 9v_3)\\ &= \alpha(0) + \beta(0)\\ &= 0 + 0 = 0 \implies \alpha\vec{u} + \beta\vec{v} \in W\end{aligned}$$

W is closed under vector addition and scalar multiplication and therefore a subspace. Note also that the equation is linear and homogeneous and hence the Superposition Principle applies, which is essentially the closure properties.

3. [2360/101922 (12 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

(a) For any matrix A, A^TA and AA^T are defined.

(b) For invertible matrices A and B, $\left|AB^{-1}\right| = |A|/|B|$.

(c) If square matrix A has 0 for an eigenvalue, then Cramer's Rule can be used to solve the system $A\vec{x} = \vec{b}$.

(d) $\operatorname{Tr}\left((2I)^3\right) = 24$ where I is the 3×3 identity matrix.

(e) $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$

(f) Homogeneous systems of linear algebraic equations are never inconsistent.

<u>**SOLUTION:**</u>

(a) **TRUE** If A is $m \times n$, then A^TA is $n \times n$ and AA^T is $m \times m$ which are both square and the determinant is defined.

(b) **TRUE** $\left|AB^{-1}\right| = |A|\left|B^{-1}\right| = |A||B|^{-1} = |A|/|B|$

(c) **FALSE** If 0 is an eigenvalue of a square matrix, then $|A| = 0$ and Cramer's Rule cannot be used.

(d) **TRUE** $\operatorname{Tr}\left((2I)^3\right) = \operatorname{Tr}\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}^3 = \operatorname{Tr}\left(\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\right) = \operatorname{Tr}\begin{bmatrix}8 & 0 & 0\\0 & 8 & 0\\0 & 0 & 8\end{bmatrix} = 24$

(e) **FALSE** $c_1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + c_2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ has no solution. $0c_1 + 0c_2$ can never equal 3.

A basis for E_7 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix} \right\}$, which has dimension 3.