



2. (15 pts) Suppose we have the series

$$s = \ln \frac{2}{3} + \ln \frac{3^2}{2 \cdot 4} + \ln \frac{4^2}{3 \cdot 5} + \ln \frac{5^2}{4 \cdot 6} + \dots$$

- (a) Find a simple expression for the partial sums  $s_n$  of the series  $s$ .  
(b) Does the series converge or diverge? Fully justify your answer. If the series converges, find its sum.

**Solution:**

- (a) To get a handle on the partial sum, let's look at the first few partial sums:

$$s_1 = \ln \frac{2}{3}$$

$$s_2 = \ln \frac{2}{3}$$

3. (15 pts) Consider the series  $\sum_{n=1}^{\infty} (-1)^n \frac{4}{(n!)^2}$ .

(a) Show that the series converges.

(b) Estimate the error in using the partial sum  $s_3$  to approximate  $s$ .

**Solution:**

(a) The series is alternating with  $b_n = \frac{4}{(n!)^2}$ . Now, clearly  $b_n$  is decreasing as the denominator grows with  $n$ .

Further,  $\lim_{n \rightarrow \infty} b_n = \frac{1}{n!} = 0$ . So by the Alternating Series Test, the

4. (25 pts) Consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n!} (n-1)!$$

- (a) Find the radius and interval of convergence for the power series  $f(x)$   
 (b) Using interval notation, for what values of  $x$  is  $f(x)$  absolutely convergent, conditionally convergent, and divergent?

**Solution:**

(a) Before we use any tests, we can simplify the factorials in the series as

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n!} (n-1)! = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n(n-1)!} (n-1)! = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n}$$

Next, applying the Ratio Test yields

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1} (x-2)^{n+1}}{4(n+1)} \frac{4n}{(-1)^n (x-2)^n} = \frac{(x-2)^n (x-2)}{(n+1)} \frac{n}{(x-2)^n} = \frac{n}{n+1} |x-2|$$

which gives a limit of

$$L = \lim_{n \rightarrow \infty} \frac{n}{n+1} |x-2| = |x-2|.$$

From the Ratio Test, we must have

$$L = |x-2| < 1 = -1 < x-2 < 1 = 1 < x < 3$$

for convergence (possibly including the endpoints). This inequality implies  $R = 1$ . Further, we almost have the full interval of convergence but we need to check the endpoints. For the left endpoint when  $x = 1$ , we have

$$f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n (1-2)^n}{4n} = \sum_{n=0}^{\infty} \frac{1}{4n}$$

which is the divergent Harmonic Series multiplied by  $1/4$ . Moving to the right endpoint when  $x = 3$ , we have

$$f(3) = \sum_{n=0}^{\infty} \frac{(-1)^n (3-2)^n}{4n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n}$$

which is the convergent Alternating Harmonic Series multiplied by  $1/4$ . Putting everything together, we find that the interval of convergence is  $(1, 2]$ .

(b) From part (a), we know the series is

$$\text{divergent for } (-\infty, 1] \cup (3, \infty).$$

Further, the right endpoint gave the Alternating Harmonic Series which is conditionally convergent since the absolute value of the series yields the regular divergent Harmonic Series. As a result,  $f(x)$  is

$$\text{conditionally convergent for } \{3\}.$$

Lastly, the Ratio Test gives absolute convergence so the from the  $L < 1$  inequality, we have

$$\text{absolute convergence for } (1, 3).$$

5. (15 pts) Starting with the Maclaurin series for  $\frac{1}{1-x}$ , write out a power series for the function below and determine its radius of convergence without the use of the Ratio or Root Tests.

$$f(x) = \frac{5}{1-4x^2}.$$

**Solution:** We know  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$  when  $|x| < 1$ . Now to find our series, we just need to replace  $x$  with  $4x^2$  in our geometric series and multiply by 5 to get

$$f(x) = \frac{5}{1-4x^2} = 5 \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 5 \cdot 4^n x^{2n}.$$

To find the radius of convergence we replace  $x$  with  $4x^2$  in the original convergence criteria to get

$$4x^2 < 1 = x^2 < \frac{1}{4} = |x| < \frac{1}{2}$$

meaning the radius of convergence is  $R = \frac{1}{2}$ .

Common Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$