

1. Determine if the series converge or diverge. Be sure to fully justify your answer and state what test that you used.

(a) (8 points) $\sum_{n=1}^{\infty} \frac{n}{3n-1}$

(b) (8 points) $\sum_{n=1}^{\infty} \frac{5}{6^{n-1}}$

(c) (8 points) $\sum_{n=1}^{\infty} \frac{5n-2n^3}{6n^3+3}$

Solution: (a) We apply the divergence test to this series:

$$\begin{aligned}\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{n}{3n-1} \\ &= \lim_{n \rightarrow \infty} \frac{n}{3n-1} \cdot \frac{1}{n}\end{aligned}$$

(c) We apply the root test to this series:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{5n - 2n^3}{6n^3 + 3}} \\
 &= \lim_{n \rightarrow \infty} \frac{5n - 2n^3}{6n^3 + 3} \\
 &= \lim_{n \rightarrow \infty} \frac{5n - 2n^3}{6n^3 + 3} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} \\
 &= \lim_{n \rightarrow \infty} \frac{\frac{5}{n^2} - 2}{6 + \frac{3}{n^3}} \\
 &= \lim_{n \rightarrow \infty} \frac{2}{6} \\
 &= \frac{1}{3} \\
 &< 1:
 \end{aligned}$$

Since $\frac{1}{3} < 1$, the series absolutely converges by the root test.

2. Determine the interval of convergence and the radius of convergence for the following power series.

(a) (15 points) $\sum_{n=1}^{\infty} \frac{(3)^n x^n}{n+1}$

(b) (15 points) $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n!}$

Solution: (a) By inspection, we see that the center of this power series is $a = 0$. We can apply the ratio test to this series to determine its radius of convergence and interval of convergence:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(3)^{n+1} x^{n+1}}{(n+1)+1} \cdot \frac{n+1}{(3)^n x^n} \\
 &= \lim_{n \rightarrow \infty} 3|x| \frac{n+1}{n+2} \\
 &= \lim_{n \rightarrow \infty} 3|x| \frac{n+1}{n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
 &= \lim_{n \rightarrow \infty} 3|x| \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} \\
 &= 3|x|:
 \end{aligned}$$

For this series to absolutely converge, we require that

$$\begin{aligned} 3|x| < 1 &\Rightarrow 1 < 3x < 1 \\ &\Rightarrow \frac{1}{3} < x < \frac{1}{3} \end{aligned}$$

From this, we see that the radius of convergence is $R = \frac{1}{3}$ and that the tentative interval of convergence is $I = \frac{1}{3}, \frac{1}{3}$

At this endpoint, the series evaluates to

$$\sum_{n=1}^{\infty} \frac{(-3)^n \frac{1}{3}^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^{\frac{1}{2}}}$$

This is an alternating series, where the positive portion of the terms are given by $b_n = \frac{1}{(n+1)^{\frac{1}{2}}}$

3. (a) (10 points) Start with the Maclaurin Series for $\frac{1}{1-x}$ to find a power series representation for $\frac{1}{1+2x^2}$. Show all work.
- (b) (8 points) Use your answer from part (a) to find its interval of convergence.

Solution:

$$(a) \frac{1}{1+2x^2} = \frac{1}{1 - (-2x^2)} \Rightarrow$$