

1. (26 pts) Evaluate the integral.

(a) $\int \frac{2x^2 - 3x + 10}{x^3 + 5x} dx$

(b) $\int \frac{1}{(x^2 - 1)^{3/2}} dx$

Solution:

(a)

$$\int \frac{2x^2 - 3x + 10}{x^3 + 5x} dx = \int \left(\frac{A}{x} + \frac{Bx + C}{x^2 + 5} \right) dx$$

Solve $A(x^2 + 5) + x(Bx + C) = 2x^2 - 3x + 10$ to find the values $A = 2$, $B = 0$, and $C = -3$.

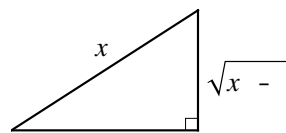
$$\int \frac{2x^2 - 3x + 10}{x^3 + 5x} dx = \int \left(\frac{2}{x} + \frac{-3}{x^2 + 5} \right) dx$$

$$= 2 \ln|x| - \frac{3}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} + C_1$$

(b) Let $x = \sec \theta$, $dx = \sec \theta \tan \theta d\theta$.

$$\begin{aligned} \int \frac{dx}{(x^2 - 1)^{3/2}} &= \int \frac{\sec \theta \tan \theta d\theta}{(\sec^2 \theta - 1)^{3/2}} = \int \frac{\sec \theta \tan \theta d\theta}{(\tan^2 \theta)^{3/2}} \\ &= \int \frac{\sec \theta \tan \theta d\theta}{\tan^3 \theta} = \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\ &= \int \frac{\cos \theta}{\sin^2 \theta} d\theta = \int \csc \theta \cot \theta d\theta = \csc \theta + C \end{aligned}$$

$$= \frac{x}{\sqrt{x^2 - 1}} + C$$



2. (20 pts) This problem has three parts.

Let $f(x) = 1 + \ln \frac{x}{x+1}$. Consider the integral $\int_1^4 f(x) dx$.

- (a) Estimate the value of the integral using the trapezoidal approximation with 3 subintervals. Fully simplify your answer by combining logarithms.
- (b) Given that $\frac{3}{4} f'''(x) < \frac{1}{50}$ for $1 \leq x \leq 4$, how large should n be to ensure that the approximation error for T_n is within 10^{-4} ? Simplify your answer.
- (c) Is the approximation found in part (a) an underestimate or overestimate? Justify your answer (it is not necessary to find the exact value of the integral.)

Solution:

(a) Let $\Delta x = \frac{b-a}{n} = \frac{3}{3} = 1$. Then

$$\begin{aligned} T_3 &= \frac{1}{2} (\Delta x) (f(1) + 2f(2) + 2f(3) + f(4)) \\ &= \frac{1}{2} (1) \left(1 + \ln \frac{1}{2} + 2 \left(1 + \ln \frac{2}{3} \right) + 2 \left(1 + \ln \frac{3}{4} \right) + 1 + \ln \frac{4}{5} \right) \\ &= \frac{1}{2} \left(6 + \ln \frac{1}{2} \cdot \frac{2^2}{3^2} \cdot \frac{3^2}{4^2} \cdot \frac{4}{5} \right) = \frac{1}{2} \left(6 + \ln \frac{1}{10} \right) \\ &= 3 - \frac{1}{2} \ln 10 \end{aligned}$$

(b) Let $K = \frac{3}{4}$, the maximum value of $f'''(x)$. Solve this inequality for n .

$$\begin{aligned} |E_{T_j}| &= \frac{K(b-a)^3}{12n^2} < 10^{-4} \\ \frac{(3-1)(3^3)}{12n^2} &< \frac{1}{10^4} \\ \frac{3^3}{4^2 n^2} &< \frac{1}{10^4} \\ n^2 &> \frac{3^3}{4^2} 10^4 \\ n &> \sqrt{\frac{3 \cdot 10^2}{4}} \\ n &> 75 \sqrt{3} \end{aligned}$$

(c) Because $f'''(x) < 0$ on $[1, 4]$, the curve $y = f(x)$ is concave down. The trapezoids all lie below the curve, so the approximation is an **underestimate**.

3. (30 pts) The following three problems are not related.

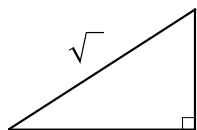
(a) Find the value of $\cot^{-1}(\cot^{-1} 1) = \cot^{-1} 1 = \frac{\pi}{5}$.

(b) Evaluate $\int_0^1 6xe^{2x} dx$. Justify any indeterminate limits.

(c) Does $\int_1^{\infty} \frac{dx}{x(1+x^5)}$ converge or diverge? Justify your answer.

Solution:

(a) Let $\theta = \cot^{-1} 1 = \frac{\pi}{5}$. Then $\cot \theta = 1 = \frac{1}{1}$. A reference triangle shows that $1=2$, so $\sin^{-1}(\cot \theta) = \sin^{-1}(1/2) = \frac{\pi}{6}$.



Note: Because $\theta > 0$, the angle is in the first quadrant.

(b) We will use integration by parts with $u = x$ and $dv = e^{2x} dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}$.

$$\begin{aligned} \int_0^1 6xe^{2x} dx &= \lim_{t \rightarrow 1^-} \int_0^t 6xe^{2x} dx \\ &= \lim_{t \rightarrow 1^-} \left[\frac{3xe^{2x}}{1} + \int_0^t \frac{3e^{2x}}{1} dx \right] \\ &= \lim_{t \rightarrow 1^-} 3xe^{2x} \end{aligned}$$

4. (24 pts) Consider the region bounded by $y = 4\sqrt{x}$, $x = 0$, and $y = 1$.

(a) Sketch and shade the region

(b) Set up but do not evaluate integrals to determine each of the following:

- I. The area of R using integration with respect to x
- II. The area of R using integration with respect to y
- III. The volume of the solid V is generated by R rotated about $y = 1$ using the disk method.

Solution:

(a) Note that the curve $y = 4\sqrt{x}$ intersects the line $y = 1$ when $4\sqrt{x} = 1 \Rightarrow x = \frac{1}{16}$.

The curve can be represented as $y = 4\sqrt{x}$.

$$(b) \quad I. \quad A = \int_0^{\frac{1}{16}} (1 - 4\sqrt{x}) \, dx$$

$$II. \quad A = \int_0^1 \frac{y^2}{16} \, dy$$

$$III. \quad V = \int_a^b \pi r^2 \, dx = \int_0^{\frac{1}{16}} \pi (1 - 4\sqrt{x})^2 \, dx$$