1. (22 points) Consider the region R in the first quadrant bounded by $y = \sin x$ and $y = \frac{2x}{}$. Set up but do not evaluate the integrals to find the following quantities:

   (a) Graph the given equations and shade the region R labeling the equations and intersection points.

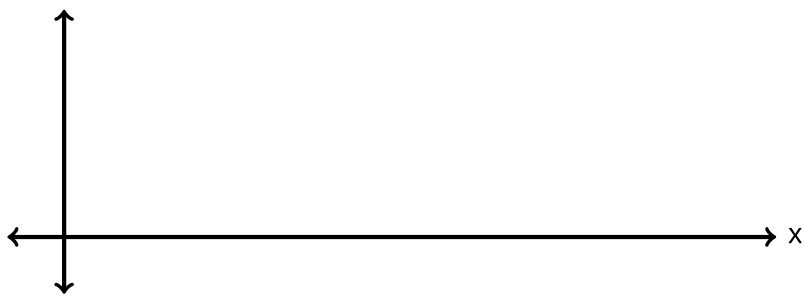
   (b) The volume of a solid with a base given by R and cross-sections perpendicular to the x-axis that are isosceles triangles with a height equal to the length of the base. (At each x, the base of the isosceles triangle is in R.)

   (c) The volume generated by rotating R about the line $x = 3$.

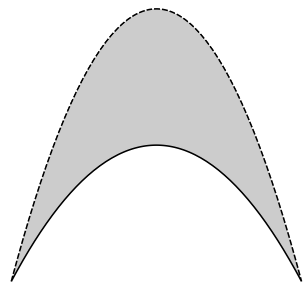
   (d) The perimeter of R.

   Solution:

   (a) Graphing the region and labeling all of the important features gives us



(a) (10 points) Find the center of mass of the region bounded between $y = 1 - x^2$ and $y = 2(1 - x^2)$ for $y > 0$. Assume a constant density ρ. The region is shown below.



(b) (8 points) Let R be the radius of the Earth. The gravitational force on a mass m at height x above the Earth's surface has magnitude $F(x) = mgR^2/(R + x)^2$. How much work is required to move the mass from $x = 0$ to a height $x = H$? (Assume $H > 0$. R, m, and g are fixed constants.)

(c) (12 points) Find the solution of the differential equation $\frac{dy}{dx} = xy\ln(x)$ with initial condition $y(1) = e$. Express your answer in the form $y = f(x)$.

Solution:

(a) By symmetry the moment about the y axis is zero so $\bar{x}$ is also zero. To find $\bar{y}$ we only need to find the moment about the x axis and the total mass. The total mass is

$$M = \int_{-1}^{1} 2(1 - x^2) - (1 - x^2)\,dx = \int_{-1}^{1} (1 - x^2)\,dx$$

Using symmetry we could also write

$$M = 2\int_{0}^{1} (1 - x^2)\,dx.$$

The total mass is

$$M = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3}.$$

The moment about the x axis is

$$M_x = \frac{1}{2}\int_{-1}^{1} \left[2(1 - x^2)\right]^2 - (1 - x^2)^2\,dx = \frac{3}{2}\int_{-1}^{1} (1 - x^2)^2\,dx = \frac{3}{2}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1} = \frac{8}{5}.$$

Finally, the coordinates of the center of mass are

$$\boxed{\bar{x} = 0;\ \bar{y} = \frac{6}{5}.}$$

(b) The total work is obtained by the integral

$$W = \int_{0}^{H} F(x)\,dx = \int_{0}^{H} \frac{mgR^2}{(R + x)^2}\,dx$$

Make the u substitution

$$u = R + x; \quad du = dx$$

which produces the integral

$$W = \int_{R}^{R+H} \frac{mgR^2}{u^2}\,du = \left[-\frac{mgR^2}{u}\right]_{u=R}^{u=R+H} = \boxed{mgR - \frac{mgR^2}{R + H} = \frac{mgHR}{R + H}.}$$

(c)

Solution:

(a) We note that $a_n = \left(1 - \frac{\ln 2}{n}\right)^n = e^{n \ln(1 - (\ln 2)/n)} = e^{\frac{\ln(1 - (\ln 2)/n)}{1/n}}$. Working with the part in the exponent, we have

$$\lim_{n\to\infty} \frac{\ln(1 - (\ln 2)/n)}{1/n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1-(\ln 2)/n} \cdot \frac{\ln 2}{n^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{-\ln 2}{1 - (\ln 2)/n} = -\ln 2$$

Putting this result into the original sequence gives

$$\lim_{n\to\infty} a_n = e^{-\ln 2} = \boxed{1/2}$$

(b) The sequence converges to 64: $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sqrt[n]{4^{2+3n}} = \lim_{n\to\infty} (4^{2+3n})^{1/n} = \lim_{n\to\infty} 4^{(2/n)+3} = \boxed{64}$

(c) The series $\sum_{n=0}^{\infty} \frac{2^{n-1} + (-1)^n}{3^n}$ is the sum of two convergent geometric series:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{2^{n-1} + (-1)^n}{3^n} &= \sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \\
&= \frac{1}{2} \cdot \frac{1}{1 - (2/3)} + \frac{1}{1 + (1/3)} \\
&= \frac{3}{2} + \frac{3}{4} = \boxed{\frac{9}{4}}
\end{aligned}$$

(d) Using the divergence test, we see:

$$\lim_{n\to\infty} \frac{n}{n}$$

(b) Before we write out a simplified, general expression for s_n, we can use partial fractions to write

$$a_n = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

Then, the sum of the first n terms of the series (including enough terms to see a pattern of cancellation) is given by

$$\begin{aligned} s_n &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \\ &= \boxed{\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}.} \end{aligned}$$

(c) To find the sum of the series, we just need to take the limit of the partial sums found in part (b). Doing so yields the convergent series

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \boxed{\frac{3}{2}.}$$