

APPM 1345

Exam 1

Spring 2023

Name

Student ID

Instructor Richard McNamara

Section 150

This exam is worth 100 points and has **4 problems**.

Make sure all of your work is written in the blank spaces provided. If your solutions do not fit, there is additional space at the end of the test. Be sure to **make a note** indicating the page number where the work is continued or it will **not** be graded.

Show all work and simplify your answers. Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.

Notes, papers, calculators, cell phones, and other electronic devices are not permitted.

End-of-Exam Checklist

1. If you finish the exam before 7:45 PM:

- Go to the designated area to scan and upload your exam to Gradescope.
- Verify that your exam has been correctly uploaded and all problems have been labeled.
- Leave the physical copy of the exam with your proctors.

2. If you finish the exam after 7:45 PM:

- Please wait in your seat until 8:00 PM.
- When instructed to do so, scan and upload your exam to Gradescope at your seat.
- Verify that your exam has been correctly uploaded and all problems have been labeled.
- Leave the physical copy of the exam with your proctors.

1. (26 pts) Parts (a) and (b) are unrelated.

(a) Find the most general form of $u(x)$ such that $u'(x) = \sec^2 x + (2 - \frac{1}{x})^2$.

Solution:

$$u'(x) = \sec^2 x + (2 - \frac{1}{x})^2 = \sec^2 x + 4 - 4x^{-1} + x^{-2}$$

$$u(x) = \tan x + 4x - \frac{4}{3}x^{3-2} + \frac{x^{-2+1}}{-1} + C$$

$$u(x) = \tan x + 4x - \frac{4}{3}x + \frac{x^{-1}}{-1} + C$$

(b) Suppose the acceleration function of a particle is given by $a(t) = 4 \cos t - 3 \sin t + 5t$, and the particle's initial velocity and position are $v(0) = 4$ and $s(0) = 5$, respectively. Find the particle's position function $s(t)$.

Solution:

$$v(t) = 4 \sin t + 3 \cos t + \frac{5t^2}{2} + C_1$$

$$v(0) = 4 = 4 \sin(0) + 3 \cos(0) + \frac{(5)(0)^2}{2} + C_1 = 0 + 3 + 0 + C_1 = 3 + C_1 \quad \Rightarrow \quad C_1 = 1$$

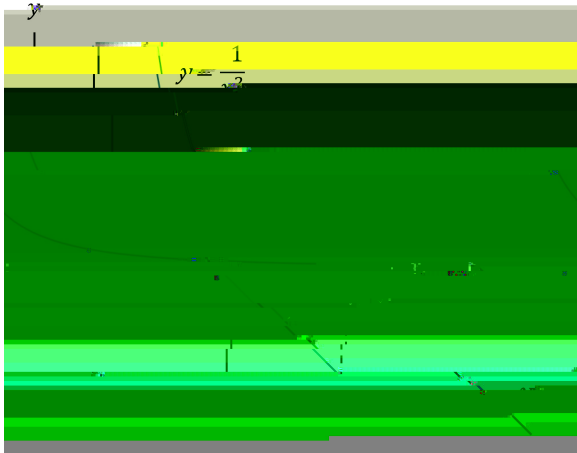
$$v(t) = 4 \sin t + 3 \cos t + \frac{5t^2}{2} + 1$$

$$s(t) = 4 \cos t + 3 \sin t + \frac{5t^3}{6} + t + C_2$$

$$s(0) = 5 = 4 \cos(0) + 3 \sin(0) + \frac{(5)(0)^3}{6} + 0 + C_2 = 4 + 0 + 0 + 0 + C_2 = 4 + C_2 \quad \Rightarrow \quad C_2 = 1$$

$$s(t) = 4 \cos t + 3 \sin t + \frac{5t^3}{6} + t + 1$$

2. (13 pts) Let the function $D(x)$ represent the vertical distance between the curve $y = 1 - x^2$ and the line $y = x$ at a given value of x . What is the minimum possible value of $D(x)$ on the interval $(0; 1)$? Use the Second Derivative Test to confirm that the result is indeed a local minimum value of the function $D(x)$.



Solution:

$$D(x) = 1 - x^2 - (x) = x^2 + x$$

$$D'(x) = 2x - 3 + 1 = x^3(2 + x^3) = \frac{x^3 - 2}{x^3}$$

$$D' = 0 \text{ for } x^3 = 2, \text{ which occurs at the critical number } x = 2^{1/3}.$$

Although D' does not exist at $x = 0$, $D(0)$ is undefined. Therefore, $x = 0$ is not a critical number of D .

$$D''(x) = 6x^{-4} = \frac{6}{x^4} > 0 \text{ for all } x \text{ in the domain of } D, \text{ and notably at the critical number } x = 2^{1/3}.$$

Therefore, the Second Derivative Test indicates that $D(x)$ has a local minimum value at $x = 2^{1/3}$. To find the actual minimum distance, evaluate $D(2^{1/3})$, as follows:

$$D(2^{1/3}) = (2^{1/3})^2 + 2^{1/3} = 2^{-2/3} + 2^{1/3} = 2^{-2/3}(1 + 2) = \boxed{(3)(2^{-2/3}) = \frac{3}{2^{2/3}} = \frac{3}{\sqrt[3]{4}}}$$

4. Parts (a) and (b) are not related.

(a) (22 pts) Let $f(x) = x^{1/3} - x^{4/3}$.

- i. Identify all critical numbers of $f(x)$.
- ii. For which values of x is $f(x)$ increasing and for which values of x is $f(x)$ decreasing? Express your answers using interval notation.
- iii. Identify the x -coordinate of each local maximum and minimum of $f(x)$ (if any). Use the First Derivative Test to classify each one.

Solution:

$$i. f'(x) = \frac{1}{3}x^{-2/3} - \frac{4}{3}x^{1/3} = \frac{1}{3}x^{-2/3}(1 - 4x) = \frac{1}{3x^{2/3}}(1 - 4x)$$

$$f' = 0 \text{ at } x = 1/4$$

f' does not exist at $x = 0$, and $x = 0$ is in the domain of f

Therefore, f has two critical numbers: $\boxed{0; 1/4}$

$$ii. \frac{1}{3x^{2/3}} > 0 \text{ for all } x \neq 0$$

$$x < 1/4: (1 - 4x) > 0 \quad ; = \text{Of}$$

$$x > 1/4: (1 - 4x) < 0$$

It follows that $f' > 0$ on $(-\infty; 0) \cup (0; 1/4)$ and $f' < 0$ on $(1/4; \infty)$.

Since f is continuous at $x = 0$ and $f' > 0$ immediately to the left of $x = 0$ and $f' > 0$ immediately to the right of $x = 0$, f has a local minimum at $x = 0$.

(b) (22 pts) Let $g(x) = 3x^5 - 5x^4$.

- i. For which values of x is $g(x)$ concave up and for which values of x is $g(x)$ concave down? Express your answers using interval notation.
- ii. Identify the x -coordinate of each inflection point of $g(x)$ (if any). Justify your answer.

Solution:

i. $g'(x) = 15x^4 - 20x^3$

$$g''(x) = 60x^3 - 60x^2 = 60x^2(x - 1)$$

$$g'' = 0 \text{ at } x = 0; 1$$

$$x^2 > 0 \text{ for all } x \neq 0$$

$$x < 1: (x - 1) < 0$$

$$x > 1: (x - 1) > 0$$

It follows that $g'' < 0$ on $(-1; 0) \cup (0; 1)$ and $g'' > 0$ on $(1; 1)$.

Therefore, g is concave down on $(-1; 0) \cup (0; 1)$ and g is concave up on $(1; 1)$

ii. Note that g is continuous on $(-1; 1)$, and in particular at $x = 0$ and $x = 1$.

g does not change concavity at $x = 0$, so there is no inflection point at $x = 0$.

g does change concavity at $x = 1$, so there is an inflection point at $x = 1$

END OF TEST

Your Initials _____

ADDITIONAL BLANK SPACE

If you write a solution here, please clearly indicate the problem number.