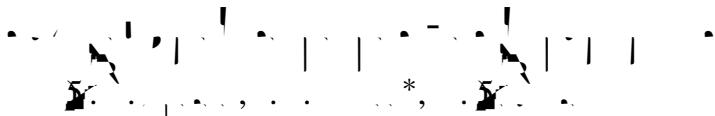




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*Department of Applied Mathematics, University of Colorado, Boulder, CO 80309-0526, USA**Received 28 April 2000; accepted 27 January 2001; available online 17 April 2001***Abstract**

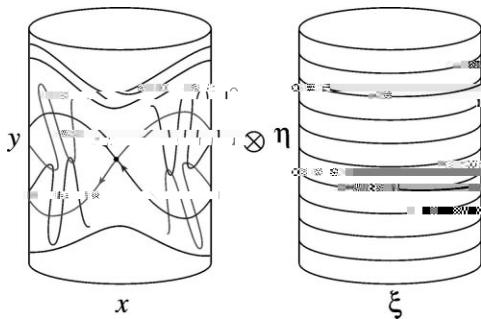
Given a function  $\Phi$  and a vector field  $\mathbf{A}$ , we study the dynamics of the flow  $\mathbf{A}_\Phi$  defined by  $\dot{\mathbf{x}} = \mathbf{A}_\Phi(\mathbf{x})$ . We prove that if  $\Phi$  is a smooth function and  $\mathbf{A}$  is a smooth vector field, then  $\mathbf{A}_\Phi$  is a smooth vector field. We also prove that if  $\Phi$  is a smooth function and  $\mathbf{A}$  is a smooth vector field, then  $\mathbf{A}_\Phi$  is a smooth vector field.

*MSC:* 37C40; 37D40; 37E50*Keywords:* Smooth function; Smooth vector field; Dynamics**1. Introduction**

Given a function  $\Phi$  and a vector field  $\mathbf{A}$ , we study the dynamics of the flow  $\mathbf{A}_\Phi$  defined by  $\dot{\mathbf{x}} = \mathbf{A}_\Phi(\mathbf{x})$ . We prove that if  $\Phi$  is a smooth function and  $\mathbf{A}$  is a smooth vector field, then  $\mathbf{A}_\Phi$  is a smooth vector field. We also prove that if  $\Phi$  is a smooth function and  $\mathbf{A}$  is a smooth vector field, then  $\mathbf{A}_\Phi$  is a smooth vector field.

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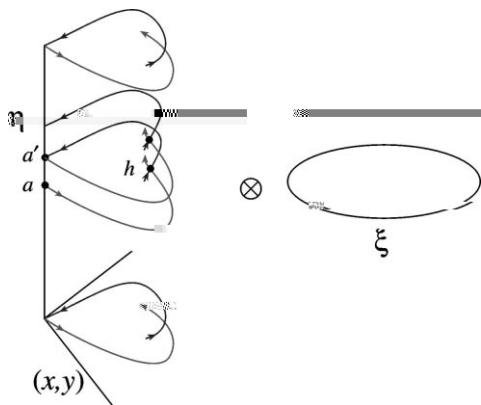
*E-mail address:* [jlucke@math.colorado.edu](mailto:jlucke@math.colorado.edu).



(1)  $\nabla \cdot h = 0$       (2)  $\nabla \times h = 0$       (3)  $\nabla^2 h = 0$

$$\begin{aligned} & \left\{ \begin{array}{l} \nabla \cdot h = 0 \\ \nabla \times h = 0 \end{array} \right. \quad \text{implies} \quad \nabla^2 h = 0. \quad \mathbf{A} = \nabla \times h \quad \text{and} \quad \nabla \cdot \mathbf{A} = 0. \\ & \nabla^2 h = 0 \quad \text{implies} \quad h = f(\xi). \quad \mathbf{A} = \nabla \times f(\xi) \quad \text{and} \quad \nabla \cdot \mathbf{A} = 0. \\ & \text{Let } \xi = T^2 \times \mathbb{R}^2 \quad \text{then} \quad h = f(\xi) = f(x, y) \quad \text{and} \quad \mathbf{A} = \nabla \times f(x, y). \end{aligned}$$

$$x' = x + y, \quad y' = -x + y, \quad y' = y - k(1 + h(x, y)), \quad x' = -kh(x, y)$$



(1) If  $\nabla \cdot h \neq 0$ , then  $\nabla \times h$  is a linear combination of  $C_a$  and  $C_{a'}$ .

$|C_a| + |C_{a'}| = |\nabla \times h|$



$$\begin{aligned} & \text{with } (x, y) \mapsto (x', y') \\ & \quad x' = x + \frac{1}{2}y, \quad y' = y + V(x). \end{aligned} \quad (4)$$

From (13),  $n=1$ ,  $V(x)=k_1(x)$ ,

### 3. A $\mathbf{t} \cdot \mathbf{a} \mathbf{b} \cdot \mathbf{t}$

$$\begin{aligned} & \mathbf{A}_{3,4,15}, \mathbf{v}_{(3)} \text{ from (3)} \\ & = *[-1.914887 - 291.4887 i, 220.187 - 17.14887 i, 1.4887 - 1.4887 i] \quad |_{87} / [116007.5716177.3568552566640] \end{aligned}$$

**4.  $\mathbf{C}_\perp$ ,  $\mathbf{t}_\perp$** 

$$\mathbf{C}_\perp = \mathbf{v} \times (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{2} \mathbf{v} \times (\mathbf{v} \times \nabla) \mathbf{v} - \frac{1}{2} \mathbf{v} \times (\mathbf{v} \cdot \nabla) \mathbf{v} - \frac{1}{2} \mathbf{v} \times (\mathbf{v} \times \nabla) \mathbf{v}$$

that  $C$  is not identically zero. Then given any  $a < b$ , there is a nonzero measure of initial states  $(v_0, v_1)$  and a sequence  $c_t \in \mathcal{C}(V)_+ \cup \mathcal{C}(V)_-$  such that the solution of (14) has momenta,  $v_t = T_2(v_{t-1}, c_t)$  satisfying  $v_0 < a$  and  $v_T > b$  for some time  $T$ .

$$\begin{aligned} & \mathbf{P}_{\pm} \cdot \mathbf{v}_0, \dots, \mathbf{v}_{t-1}, \mathbf{v}_t, \dots, \mathbf{v}_T, c_- \in \mathcal{C}(V)_-, c_+ \in \mathcal{C}(V)_+, \dots, t, \dots, x_t = c_{\pm} \cdot \mathbf{v}_t, C(t) = \pm 1. \\ & (14) \quad \tilde{L}(v, v') = T(v, v') + W(v) + \tilde{C}(v), \\ & \tilde{C} = V(c_{\pm}(v)) - C(v) \geq 0. \end{aligned}$$

$$\text{If } \tilde{C}(v+2) - \tilde{C}(v) > 0, \quad \tilde{L}(v+2, v'+2) - \tilde{L}(v, v') \leq 0. \quad \mathbf{A} \cdot \mathbf{v} \cdot \mathbf{v}' \leq 0. \quad \square$$

#### 4.2. Standard example

$$\begin{aligned} & L(x, x', v, v') = \frac{1}{2} (x' - x)^2 + \frac{1}{2} (v' - v)^2 + k \cdot x(1 + h(v)), \\ & \text{where } k > 0, \quad h > 0, \quad \mathbf{A} \cdot \mathbf{v} \cdot \mathbf{v}' \leq 0, \quad (1), \dots, (n-1), \quad (15). \end{aligned}$$

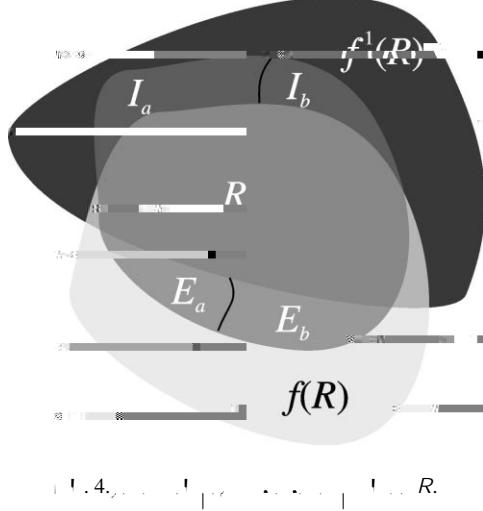
$$\begin{aligned}
 & t \mapsto (\cdot, \cdot)^* = 2(m\pi, \cdot, \cdot, (0, 2m)) \text{ for } (-2m, \cdot), \\
 & (\cdot, \cdot)^* = (2m+1), \quad \text{for } (\cdot, \cdot, \cdot, (0, 2m)), \\
 & \text{and } (\cdot, \cdot)^* = 0, \quad \text{for } (\cdot, \cdot, \cdot, (0, 0)). \quad (16)
 \end{aligned}$$

$\mathbf{A} = \frac{1}{T} \sum_{t=0}^{T-1} U'(\cdot, \cdot, \cdot, \cdot)$ ,  $g^* = \frac{1}{T} \sum_{t=0}^{T-1} g(\cdot, \cdot, \cdot, \cdot)$ ,  $g^* - g = \frac{1}{T} \sum_{t=0}^{T-1} (U'(\cdot, \cdot, \cdot, \cdot) - g(\cdot, \cdot, \cdot, \cdot))$ ,  $(\cdot, \cdot)^* = \frac{1}{T} \sum_{t=0}^{T-1} (\cdot, \cdot)^*$ ,  $\mathbb{T}^2 = \{(\cdot, \cdot) : 0 < \cdot, \cdot < 2\}$ .

$$(\cdot, \cdot)^* = (\cdot_{t+1} - \cdot_t)^* = \frac{1}{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} (U'(\cdot_t)).$$

$$\mathbf{A} = \frac{1}{T} \sum_{t=0}^{T-1} U'(\cdot, \cdot, \cdot, \cdot), \quad g^* = \frac{1}{T} \sum_{t=0}^{T-1} g(\cdot, \cdot, \cdot, \cdot).$$

$$\langle \cdot, \cdot \rangle = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} U'(\cdot, \cdot, \cdot, \cdot) d\mu.$$



$E = \{z \in R : f(z) \notin R\} = R \setminus f^{-1}(R)$ .

$$\mu(E) = \mu(R \setminus f^{-1}(R)) = \mu(R) - \mu(R \cap f^{-1}(R)) = \mu(R) - \mu(f(R) \cap R) = \mu(R \setminus f(R)) = \mu(I). \quad (17)$$

$S^t = f(S^{t-1}) \cap R$  stays in  $R$  for  $t < t^*$ , while  $f(S^{t-1}) \cap R$  is outside  $R$  for  $t \geq t^*$ .  
 $S^0 = I$ ,  $S^t = f(S^{t-1}) \cap R = f(S^{t-1} \setminus E)$ .

$\sum_{j=0}^{\infty} \mu(f^{-j}(S^t)) \leq \mu(R)$ ,  $\forall t \in \mathbb{N}$ .  
 $\lim_{t \rightarrow \infty} \mu(S^t) \rightarrow 0$ ,  $\forall t \in \mathbb{N}$ .  
 $\lim_{t \rightarrow \infty} f^{-j}(S^t) \cap R \rightarrow E$ .

$$\mu(p(I_a) \cap E_b) = \mu(p(I_a)) - \mu(p(I_a) \cap E_a) \geq \mu(I_a) - \mu(E_a).$$

□

**A.4.**  $I_t = R \setminus f_t(R)$ ,  $E_t = R \setminus f_t^{-1}(R)$ .

$$(17) \quad S_k^t = I_{k-1}, \quad S_k^{t+1} = f_t(S_k^t \setminus E_t), \quad k \leq t$$

**A.5.**  $\mu(S_k^t) < \mu(R)$

$$(18) \quad \sum_{k=-\infty}^t \mu(S_k^t) < \mu(R)$$

**Lemma 4.** Let  $f_t$  be a sequence of measure-preserving homeomorphisms, and  $R$  a measurable set with incoming sets  $I_t$  and exit sets  $E_t$ .

### 5.2. Maps of the cylinder

$$\begin{aligned} & \text{Cylinder } C \text{ with boundary } f(C) \\ & = \frac{y'}{C} x' - y, x. \end{aligned}$$

$$\begin{aligned} & \text{Cylinder } C \text{ with boundary } f(C) \\ & \text{A shaded region } U \subset T, \\ & U = \{z \in T : f^{-1}(z) \in B\}. \end{aligned}$$

$$\begin{aligned} & \text{D shaded region } D \subset B, \\ & D = \{z \in B : f^{-1}(z) \in T\}. \end{aligned}$$

$$\begin{aligned} & \text{A shaded region } \mu(U) - \mu(D) = \dots \\ & \text{A shaded region } \frac{1}{4} \dots \end{aligned}$$

**Corollary 5.** Suppose that  $f_t$  is a sequence of area and end-preserving homeomorphisms of the cylinder, and that the net flux  $\int_{t=0}^{\infty} \phi_t \cdot n dt > 0$ . Let  $A$  denote the annulus bounded by the circles  $\{y = a\}$  and  $\{y = b\}$  where  $a < b$ . Then, there is a set of positive measure of orbits that cross the annulus.

**Proof.**  $U_t(a) \cup D_t(a)$  is shaded.  $f_t^{-1}(D_t(a)) \cup f_t^{-1}(U_t(b))$  is shaded.  $\mu(U_t(b)) \geq \dots > 0$ .  $E_t = f_t^{-1}(U_t(b))$ .  $\square$

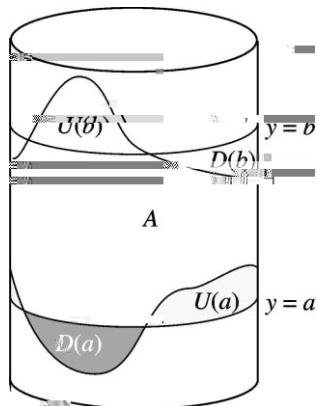


Fig. 5. A cylinder representing the annulus  $A$ .

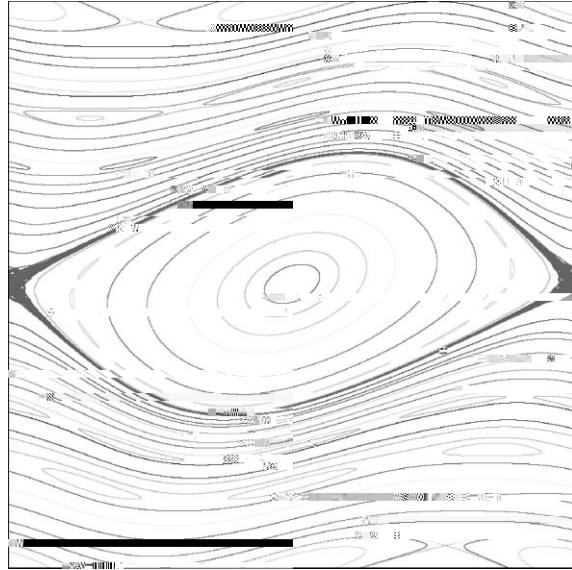


Fig. 6. A contour plot of a standard map with  $k = 0.5$ . The horizontal bar indicates the width of the central island.

### 5.3. Standard map with net flux

Let  $V(x, y)$  be a function defined on  $(0, 2\pi) \times \mathbb{R}$ . Then  $\nabla V(x, y) = V(2\pi) - V(0)$ .  
 $x' = x + y, \quad y' = y - kV(x) + \frac{1}{2}V'(x)$ .  
 $y = 0, \quad k < k_{cr} \approx 0.971635406$ .  
 $y = 2\pi, \quad k = 0.5$ .  
 $f(x, y+2\pi m) = f(x, y) + \frac{1}{2}V'(m, m)$ .  
 $x = \frac{1}{2}V(0) - \frac{1}{2}V(2\pi)$ .  
 $y = 2\pi - k$ .

## 6. Procedure

Let  $(x_{t-1}, y_{t-1})$  be the initial point. Then  $z_{t-1} = (x_{t-1}, y_{t-1}, t)$ .  
 $(12)$  is used to calculate  $(x_t, y_t)$ .

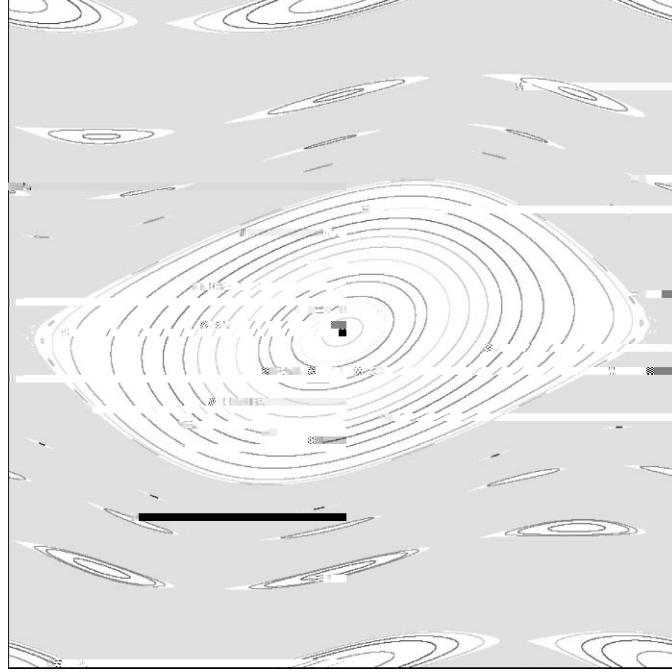


Fig. 7. A 3D plot of the function  $\psi$ , given by (18), with  $k = 0.5$ ,  $a = 0.1$ , and  $\mathcal{F} = 4^{-2}/1000$ . The plot shows a complex, multi-layered surface with numerous small peaks and valleys, resembling a topographic map or a contour plot of a function.

$$\begin{aligned} z_t &= (z_{t-1}) = \frac{x_t}{-x_{t-1} + 2x_t + \frac{1}{t} V(x_t) 1 + C(t)} \\ &\quad - t_{t-1} + 2 t + W(t) + V(x_t) - C(t) \end{aligned} \quad (19)$$

Let  $x_t = c_t \in \mathcal{V}$  and  $x_{t-1} = c_{t-1} \in \mathcal{V}$ . Then  $x_t = c_t \in \mathcal{V}$  if and only if  $c_t = 0$ . Let  $x_t = c_t \in \mathcal{V}$  and  $x_{t-1} = c_{t-1} \in \mathcal{V}$ . Then  $x_t = c_t \in \mathcal{V}$  if and only if  $c_t = 0$ .

**Lemma 6.** Suppose that  $\psi$ , given by (19), is a  $C^2$  map of  $\mathbb{T}^4$ , such that  $1 + C(\cdot) \geq \alpha > 0$ . Then, for any sequence  $\{c_0, c_1, \dots\}$  with  $c_t \in \mathcal{V} \cap \mathcal{A}$ , any initial condition  $(x_0, c_0)$ , and any  $\epsilon > 0$ , there exists an orbit  $z_t = (x_t, x_{t+1}, c_t, c_{t+1})$ ,  $t \geq 0$  of  $\psi$  such that

$$|x_t - c_t| \leq \epsilon, \quad t \geq 0,$$

provided

$$0 \leq \alpha < \alpha_0 = \frac{1}{(4 + a)}, \quad (20)$$

where  $\alpha(b) \equiv \inf_{t \geq 0} |V(c_t \pm b)|$ .



$$\sum_{t=1}^{\infty} V(c_{t+1}^{(+)}) > (4+a) \cdot \frac{a}{1-a} \cdot \frac{1+C(\epsilon)}{1-\epsilon} \geq \frac{4+a}{1-a} > 4+a. \quad (20)$$

$\square$

$S = \mathbb{R}^2 \times (0, 1) \cap W_0$ , where  $W_0 = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$ .  $T = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1, x^2 + y^2 \leq 1\}$ .  $B_t = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1, x^2 + y^2 \leq t^2\}$ ,  $t \geq 1$ .  $S \subset T \subset B_1 \subset \dots \subset B_t \subset \dots$ .  $U_t = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1, x^2 + y^2 \leq t^2, x^2 + y^2 \geq t^2 - \epsilon^2\}$ ,  $t \geq 1$ .  $S \subset U_1 \subset \dots \subset U_t \subset \dots$ .  $W_t = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1, x^2 + y^2 \leq t^2, x^2 + y^2 \geq t^2 - \epsilon^2, x^2 + y^2 \leq t^2 + \epsilon^2\}$ ,  $t \geq 1$ .  $S \subset W_1 \subset \dots \subset W_t \subset \dots$ .

$$\begin{aligned} \left| \frac{\partial}{\partial t} \left( \frac{1}{2} M^2 r^{2t} \right) \right| &\leq r^t, \quad r > 1, \quad \text{and} \quad r^2 - wr - 1 = 0, \quad W = \int_x (2 + |W(x)|). \\ \left| \frac{\partial}{\partial t} \left( \frac{1}{2} M^2 r^{2t} \right) \right| &\leq \frac{1}{2} M^2 r^{2t}. \end{aligned}$$

$$\begin{aligned} \sum_{t=0}^T \left| \frac{\partial}{\partial t} \left( \frac{1}{2} M^2 r^{2t} \right) \right| &\leq \sum_{t=0}^T r^t, \quad W = 0, \quad \text{and} \quad \left| \frac{\partial}{\partial t} \left( \frac{1}{2} M^2 r^{2t} \right) \right| \leq \frac{1}{2} M^2 r^{2t}, \quad t \leq T. \end{aligned} \quad \square$$

**Résumé.** L'objectif de cet article est d'étudier les propriétés des solutions d'un système différentiel à temps continu et discréte. Les résultats sont appliqués à la modélisation de l'écoulement d'un fluide dans un canal rectangulaire.

### 6.1. Standard example, continued

$$\begin{aligned} \text{Exemple 7.} &\quad \text{Soit } V(x) = kx, \quad C(t) = h, \quad h \in [0, 1], \quad \text{et} \quad a = 2. \quad \text{Alors} \quad \frac{k(1-h)}{4+2} \leq 0. \end{aligned}$$

$$\text{Exemple 7.} \quad M = kh, \quad W = 1 \# DB)DB \quad b B Db \quad b b \# ET bDD B$$

where  $h < 1$  and  $\|v_0\| \leq 0$ ,  $\|v_1\| \leq \dots \leq \|v_n\| \leq \dots \leq \|v_k\|$ . Then  $\|v_{k+1}\| \leq \dots \leq \|v_n\| \leq \dots \leq \|v_k\| \leq \dots \leq \|v_1\|$ .

## 7. Conclusion

We have shown that the two-dimensional discrete dynamical system associated with the difference equation (1) exhibits a wide range of behaviors. For  $a > 0$  and  $b > 0$  there is a unique fixed point at the origin which is unstable. For  $a < 0$  and  $b > 0$  there is a unique stable fixed point at the origin. For  $a < 0$  and  $b < 0$  the behavior depends upon the initial conditions. For some initial conditions there is an unstable fixed point at the origin, while for other initial conditions there is an unstable periodic orbit. For  $a > 0$  and  $b < 0$  there is an unstable fixed point at the origin and an unstable limit cycle. We also show that there is a wide range of values of  $a$  and  $b$  for which the system exhibits a chaotic behavior. The analysis is based on the computation of the Lyapunov exponents. The two-dimensional dynamical system is related to a one-dimensional discrete dynamical system. We have also shown that the two-dimensional system can be reduced to a one-dimensional dynamical system which is related to the difference equation (1). Finally, we have shown that the two-dimensional dynamical system can be reduced to a one-dimensional dynamical system which is related to the difference equation (1). This one-dimensional dynamical system is related to the difference equation (1). The one-dimensional dynamical system has been studied by several authors (e.g., [1,2,16,17]).

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